Math 175
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Section 2.8: Onto Linear Transformations

Today we are going to learn about **onto** linear transformations.

Definition: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if the range of T is $\mathbb{R}^m$, that is, if, for every $\mathbf{b} \in \mathbb{R}^m$, there exists a $\mathbf{v} \in \mathbb{R}^n$ such that $T(\mathbf{v}) = \mathbf{b}$.

We have already noted that for every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, there exists an $m \times n$ matrix A such that $T(x) = Ax$. Thus, deciding whether or not a linear transformation is onto, i.e., if $T(x) = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$, is equivalent to deciding whether or not $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$. We know the answer to this question! We learned it in Section 1.6.

1. Complete the following: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is <u>onto</u> if and only if

   (a) $A\mathbf{x} = \mathbf{b}$ has __________ for every $\mathbf{b} \in \mathbb{R}^m$

   (b) rank $A =$ _______

   (c) Columns of A span _______

   (d) there is a pivot position in every _______ of A

   (e) rref(A) has ____________________

2. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \mathbf{T}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 4x_2 \\ 2x_1 + x_3 \end{bmatrix}$.

   (a) Find the standard matrix A for T.

   (b) Is T onto? Why or why not?

   (c) Can you find a vector $\mathbf{v} \in \mathbb{R}^3$ with $T(\mathbf{v}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$? If so, find such a vector $\mathbf{v}$; if not, explain why not.

3. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = \mathbf{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix}$.

(a) Find the standard matrix A for T.

(b) Is T onto? Why or why not?

(c) Can you find a vector $\mathbf{v} \in \mathbb{R}^2$ with $T(\mathbf{v}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$? If so, find such a vector $\mathbf{v}$; if not, explain why not.

4. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Mark each of the following **True or False** and explain.

(a) If $m > n$, then T is not onto.

(b) If $m \leq n$, then T is onto.