Section 4.1: Subspaces

Definition: A set W of vectors from $\mathbb{R}^n$ is a **subspace** of $\mathbb{R}^n$ if

1. $\mathbf{0} \in W$,
2. if $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$ (we say that W is **closed under vector addition**), and
3. if $\mathbf{u} \in W$ and c is any scalar, then $c\mathbf{u} \in W$ (we say that W is **closed under scalar multiplication**).

In this handout, we are going to decide whether a given set is a subspace of $\mathbb{R}^n$ as well as learn about two special subspaces associated with an $m \times n$ matrix A. We will also find minimal generating sets for the given set if it is a subspace.

1. Is $W = \{\mathbf{0}\}$ a subspace of $\mathbb{R}^n$? Why or why not?

2. Is $W = \mathbb{R}^n$ a subspace of $\mathbb{R}^n$? Why or why not?

3. Let $\mathbf{v}_1, \mathbf{v}_2$ be vectors in $\mathbb{R}^n$. Answer the following questions to decide if $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a subspace of $\mathbb{R}^n$.

   (a) Is $\mathbf{0} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$? Explain.

   (b) If $\mathbf{v}, \mathbf{w} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, is $\mathbf{v} + \mathbf{w} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$? Explain.

   (c) If $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and c is a scalar, is $c\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$? Explain.

(d) Is $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ a subspace? Why or why not?

4. Based on your work in the previous question, do you think $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k\}$ is a subspace of $\mathbb{R}^n$ for vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k$ of $\mathbb{R}^n$? Explain.

5. Consider the set $W = \left\{ \begin{bmatrix} 5s - 3t \\ -7s + 8t \\ 2s \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$ of vectors from $\mathbb{R}^3$. We wish to decide if W is a subspace of $\mathbb{R}^3$.

(a) Is $\mathbf{0} \in W$? Explain.

(b) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in W. Then $\mathbf{u} = \begin{bmatrix} 5s_1 - 3t_1 \\ -7s_1 + 8t_1 \\ 2s_1 \end{bmatrix}$ for some $s_1, t_1 \in \mathbb{R}$ and $\mathbf{v} = \begin{bmatrix} 5s_2 - 3t_2 \\ -7s_2 + 8t_2 \\ 2s_2 \end{bmatrix}$ for some $s_2, t_2 \in \mathbb{R}$. Determine $\mathbf{u}+\mathbf{v}$ and decide, with explanation, if $\mathbf{u}+\mathbf{v} \in W$.

(c) Let $\mathbf{u}$ be a vector in W and c be a scalar. Is $c\mathbf{u} \in W$? Explain.

(d) Do you think W is a subspace of $\mathbb{R}^3$? Explain.

6. Consider the set $W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \mid w_1 - w_2 + w_3 = 0 \right\}$.

(a) Is $\mathbf{0} \in W$? Explain.

(b) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in W. Then $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ for some $u_1, u_2, u_3 \in \mathbb{R}$ with $u_1 - u_2 + u_3 = 0$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ for some $v_1, v_2, v_3 \in \mathbb{R}$ with $v_1 - v_2 + v_3 = 0$. Determine, with explanation, if $\mathbf{u} + \mathbf{v} \in W$.

(c) Let $\mathbf{u}$ be a vector in W and c be a scalar. Is $c\mathbf{u} \in W$? Explain.

(d) Do you think W is a subspace of $\mathbb{R}^3$? Explain.

7. Consider the set $W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \mid w_1 - w_2 + w_3 = 2 \right\}$. Is W a subspace of $\mathbb{R}^3$? Explain.

Now we want to discuss two subspaces associated with a given $m \times n$ matrix A, the **column space** and **null space** of A.

The **column space** of A, denoted Col A, is the span of the columns of A, i.e., Col $A = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ where $\mathbf{a}_i$ is the ith column of A.

The **null space** of A, denoted Null A, is the set of solutions to the equation $A\mathbf{x} = \mathbf{0}$, that is, Null $A = \{\mathbf{x} \in \mathbb{R}^n \mid Ax = \mathbf{0}\}$.

8. Let A be an $m \times n$ matrix. Clearly, Col A is a subspace since it's the span of a set of vectors, but is Null A a subspace? We will answer that question next but before we do, Col A is a subspace of _______ and Null A is a set of vectors from _______.

9. Let A be an $m \times n$ matrix. Consider the set Null $A = \{\mathbf{x} \in \mathbb{R}^n \mid Ax = \mathbf{0}\}$.

(a) Is $\mathbf{0} \in$ Null A? Explain.

(b) If $\mathbf{u}, \mathbf{v} \in$ Null A, is $\mathbf{u} + \mathbf{v} \in$ Null A? Explain.

(c) If $\mathbf{u} \in$ Null A and c is a scalar, is $c\mathbf{u} \in$ Null A? Explain.

(d) Is Null A a subspace? Explain.

10. Let $A = \begin{bmatrix} 1 & 3 & -1 \\ -3 & 5 & 7 \end{bmatrix}$.

(a) Col A is a subspace of _______ and Null A is a subspace of _______.

(b) Find a minimal generating set for Col A (Hint: the columns of A are a generating set but do you need all of them?)

(c) Find a minimal generating set of Null A. (Hint: start by finding the general solution to $A\mathbf{x} = \mathbf{0}$.)

11. Let's reconsider some of our previous examples.

(a) We have shown that the set $W = \left\{ \begin{bmatrix} 5s - 3t \\ -7s + 8t \\ 2s \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

i. Can you find a generating set for W? Is your set minimal?

ii. Find a matrix A such that $W = \text{Col } A$.

(b) Consider the subspace $W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \mid w_1 - w_2 + w_3 = 0 \right\}$ of $\mathbb{R}^3$.

i. Find a matrix A such that $W = \text{Null } A$.

ii. Find a minimal generating set for W.

iii. Why did we choose to find A such that $W = \text{Null } A$ rather than $W = \text{Col } A$?