Math 175
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Section 7.1 Vector Spaces

Definition: A (real) **vector space** is a set V of objects we call **vectors** with two operations: **vector addition** $\oplus$ and **scalar multiplication** $\odot$ such that

(a) $u \oplus v \in V$ for all vectors $u, v \in V$

(b) $c \odot u \in V$ for all scalars $c \in \mathbb{R}$ and vectors $u \in V$

(1) $u \oplus v = v \oplus u$ for all vectors $u, v \in V$

(2) $(u \oplus v) \oplus w = u \oplus (v \oplus w)$ for all vectors $u, v, w \in V$

(3) there exists $\mathbf{0}_V \in V$ such that $u \oplus \mathbf{0}_V = u$ for all vectors $u \in V$

(4) for all vectors $u \in V$, there exists $-u \in V$ such that $u \oplus -u = -u \oplus u = \mathbf{0_V}$

(5) $1 \odot u = u$ for all vectors $u \in V$

(6) $(ab) \odot u = a \odot (b \odot u)$ for all scalars $a, b \in \mathbb{R}$ and vectors $u \in V$

(7) $a \odot (u \oplus v) = (a \odot u) \oplus (a \odot v)$ for all scalars $a \in \mathbb{R}$ and vectors $u, v \in V$

(8) $(a + b) \odot u = (a \odot u) \oplus (b \odot u)$ for all scalars $a, b \in \mathbb{R}$ and vectors $u \in V$

1. Verify that $\mathbb{R}^n$ together with ordinary vector addition and scalar multiplication is a vector space:

   (a) $\mathbf{u} + \mathbf{v} \in \mathbb{R}^n$ for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ since ...

   (b) $c\mathbf{u} \in \mathbb{R}^n$ for all scalars $c \in \mathbb{R}$ and vectors $\mathbf{u} \in \mathbb{R}^n$ since ...

   (1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$
   Theorem _______ _______

   (2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$
   Theorem _______ _______

   (3) there exists $\mathbf{0}_{\mathbb{R}^n} \in V$ such that $\mathbf{u} + \mathbf{0}_{\mathbb{R}^n} = \mathbf{u}$ for all vectors $\mathbf{u} \in \mathbb{R}^n$
   $\mathbf{0}_{\mathbb{R}^n} =$ __________; Theorem _______ _______

   (4) for all vectors $\mathbf{u} \in \mathbb{R}^n$, there exists $-\mathbf{u} \in \mathbb{R}^n$ such that $\mathbf{u} + -\mathbf{u} = \mathbf{0}_{\mathbb{R}^n}$
   Theorem _______ _______

   (5) $1\mathbf{u} = \mathbf{u}$ for all vectors $\mathbf{u} \in \mathbb{R}^n$ since ...

(6) $(ab)\mathbf{u} = a(b\mathbf{u})$ for all scalars $a, b \in \mathbb{R}$ and vectors $\mathbf{u} \in \mathbb{R}^n$

Theorem _______ _______

(7) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ for all scalars $a \in \mathbb{R}$ and vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

Theorem _______ _______

(8) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ for all scalars $a, b \in \mathbb{R}$ and vectors $\mathbf{u} \in \mathbb{R}^n$

Theorem _______ _______

2. Verify that $\mathcal{M}_{m\times n} = \{A \mid A \text{ is an } m \times n \text{ matrix}\}$ together with ordinary matrix addition and scalar multiplication is a vector space:

(a) $A + B \in \mathcal{M}_{m\times n}$ for all "vectors" $A, B \in \mathcal{M}_{m\times n}$ since ...

(b) $cA \in \mathcal{M}_{m\times n}$ for all scalars $c \in \mathbb{R}$ and "vectors" $A \in \mathcal{M}_{m\times n}$ since ...

(1) $A + B = B + A$ for all "vectors" $A, B \in \mathcal{M}_{m\times n}$

Theorem _______ _______

(2) $(A + B) + C = A + (B + C)$ for all "vectors" $A, B, C \in \mathcal{M}_{m\times n}$

Theorem _______ _______

(3) there exists $\mathbf{0}_{\mathcal{M}_{m\times n}} \in \mathcal{M}_{m\times n}$ such that $A + \mathbf{0}_{\mathcal{M}_{m\times n}} = A$ for all "vectors" $A \in \mathcal{M}_{m\times n}$

$\mathbf{0}_{\mathcal{M}_{m\times n}} =$ __________; Theorem _______ _______

(4) for all "vectors" $A \in \mathcal{M}_{m\times n}$, there exists $-A \in \mathcal{M}_{m\times n}$ such that $A + -A = \mathbf{0}_{\mathcal{M}_{m\times n}}$

Theorem _______ _______

(5) $1A = A$ for all "vectors" $A \in \mathcal{M}_{m\times n}$ since ...

(6) $(st)A = s(tA)$ for all scalars $s, t \in \mathbb{R}$ and "vectors" $A \in \mathcal{M}_{m\times n}$

Theorem _______ _______

(7) $s(A + B) = sA + sB$ for all scalars $s \in \mathbb{R}$ and vectors $A, B \in \mathcal{M}_{m\times n}$

Theorem _______ _______

(8) $(s + t)A = sA + tA$ for all scalars $s, t \in \mathbb{R}$ and "vectors" $A \in \mathcal{M}_{m\times n}$

Theorem _______ _______

3. Suppose $S = \mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$. Define two operations on S by $(a, b) \oplus (c, d) = (a + c, 0)$ and $k \odot (a, b) = (ka, kb)$. It turns out that $(S, \oplus, \odot)$ is **not** a vector space. Find an axiom that fails.

4. Suppose $S = \mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$. Define two operations on S by $(a, b) \oplus (c, d) = (a + c, b + d)$ and $k \odot (a, b) = (ka, b)$. It turns out that $(S, \oplus, \odot)$ is **not** a vector space. Find an axiom that fails.

5. Suppose $S = \mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$. Define two operations on S by $(a, b) \oplus (c, d) = (a+c+1, b+d+1)$ and $k \odot (a, b) = (ka, kb)$. It turns out that $(S, \oplus, \odot)$ is **not** a vector space. Find an axiom that fails.

6. Is $S = \left\{ \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ with ordinary matrix addition and ordinary scalar multiplication a vector space? Why or why not?

Two other examples of vectors spaces:

- $\mathcal{P} = \{p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \in \mathbb{R}\}$, the set of all polynomials with addition and scalar multiplication defined as follows: if $p(x), q(x) \in \mathcal{P}$ and $c \in \mathbb{R}$, then

$$\begin{aligned} p(x) \oplus q(x) &= (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) + (b_0 + b_1x + b_2x^2 + \cdots + b_nx^n) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n \end{aligned}$$

  and $c \odot p(x) = c(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = (ca_0) + (ca_1)x + (ca_2)x^2 + \cdots + (ca_n)x^n$.
- $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = \{T : \mathbb{R}^n \to \mathbb{R}^m \mid T \text{ is a linear transformation}\}$ with addition and scalar multiplication is defined as follows: if $U, T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $c \in \mathbb{R}$, then $U \oplus T$ is the linear transformation defined by $(U \oplus T)(v) = U(v) + T(v)$, and $c \odot T$ is the linear transformation defined by $(c \odot T)(v) = cT(v)$.

7. What is $\mathbf{0}_{\mathcal{P}}$?

8. $\mathbf{0}_{\mathcal{L}}$?

There are many important facts that we can derive from the axioms of a vector space. Prove these facts by filling in the blanks below.

Theorem 7.2 Let $(V, \oplus, \odot)$ be a vector space. Then for all $u, v, w \in V$ and scalars $a \in \mathbb{R}$, the following statements are true:

(a) $u \oplus v = w \oplus v$ then $u = w$

(b) $u \oplus v = u \oplus w$ then $v = w$

(c) $\mathbf{0}_V$ is unique

(d) For every $v \in V$, there is a unique $-v \in V$ such that $v \oplus -v = \mathbf{0}_V$

(e) $0 \odot v = \mathbf{0}_V$

(f) $a \odot \mathbf{0}_V = \mathbf{0}_V$

(g) $-1 \odot v = -v$

(h) $(-a) \odot v = a \odot (-v) = -(a \odot v)$

Proof.

(a) Suppose $u \oplus v = w \oplus v$.

Then $(u \oplus v) \oplus -v = (w \oplus v) \oplus -v$.

So $u \oplus (v \oplus -v) = w \oplus (v \oplus -v)$ by Axiom ___ of a vector space.

Then $u \oplus (\mathbf{0}_V) = w \oplus (\mathbf{0}_V)$ by Axiom ___ of a vector space.

So $u = w$ by Axiom ___ of a vector space.

(b) Suppose $u \oplus v = u \oplus w$.

(c) We want to show $\mathbf{0}_V$ is unique. The property for $\mathbf{0_V}$ is that $u \oplus \mathbf{0_V} = u$ for all $u \in V$. So, suppose there are two such vectors, say $\mathbf{0_V}$ and $\mathbf{0'_V}$ are **two** vectors in V such that $u \oplus \mathbf{0_V} = u$ and $u \oplus \mathbf{0'_V} = u$ for all vectors $u \in V$.

Then, with $u = \mathbf{0_V}$ and using $\mathbf{0'_V}$ we have $\mathbf{0_V} \oplus \mathbf{0'_V} =$ _______.

Also, with $u = \mathbf{0'_V}$ and using $\mathbf{0_V}$ we have $\mathbf{0'_V} \oplus \mathbf{0_V} =$ _______.

Now, since $\mathbf{0_V} \oplus \mathbf{0'_V} = \mathbf{0'_V} \oplus \mathbf{0_V}$ by Axiom ___ of a vector space, we have ___ = ___.

Thus, $\mathbf{0}_V$ is unique.

(d) Now we want to show that for every $v \in V$, $-v \in V$ unique. The property for $-v$ is that $v \oplus -v = \mathbf{0_V}$. Let $v \in V$. The property for $-v$ is that $v \oplus -v = \mathbf{0_V}$. Suppose there are two vectors $-v, w \in V$ such that $v \oplus -v = \mathbf{0}_V$ and $v \oplus w = \mathbf{0}_V$.

So, __________ = __________.

Using part ___, we get $-v = w$ and thus $-v$ is unique.

(e) Now we show $0 \odot v = \mathbf{0}_V$ for every $v \in V$. Let $v \in V$. Now

$$\begin{aligned}
(0 \odot v) \oplus (0 \odot v) &= (_______) \odot v \text{ by Axiom } ___ \text{ of a vector space} \\
&= ___ \odot v \\
(0 \odot v) \oplus (0 \odot v) &= (0 \odot v) \oplus \mathbf{0}_V \text{ by Axiom } ___ \text{ of a vector space} \\
0 \odot v &= \mathbf{0}_V \text{ by part } ___
\end{aligned}$$

(f) Now we show $a \odot \mathbf{0}_V = \mathbf{0}_V$ for every scalar a. Let a be a scalar. Consider

$$\begin{aligned}
a \odot \mathbf{0}_V &= a \odot (___ \oplus ___) \text{ by Axiom } ___ \text{ of a vector space} \\
&= _____ \oplus _____ \text{ by Axiom } ___ \text{ of a vector space} \\
a \odot \mathbf{0}_V \oplus \mathbf{0}_\mathbf{V} &= a \odot \mathbf{0}_V \oplus a \odot \mathbf{0}_V \text{ by Axiom } ___ \text{ of a vector space}
\end{aligned}$$

So $\mathbf{0}_V = a \odot \mathbf{0}_V$ by part ___.

(g) Next, we show $-1 \odot v = -v$ for every vector $v \in V$. So

$$\begin{aligned}
v \oplus (-1) \odot v &= (___ \odot v) \oplus (-1 \odot v) \text{ by Axiom } ___ \text{ of a vector space} \\
&= (___ + ___) \odot v \text{ by Axiom } ___ \text{ of a vector space} \\
&= _____ \\
&= \mathbf{0}_V \text{ by part } ___
\end{aligned}$$

Since $-v$ is unique by part ___ and $v \oplus (-1) \odot v = \mathbf{0}_\mathbf{V}$, we have $-1 \odot v = -v$.

(h) Now we show $(-a) \odot v = a \odot (-v) = -(a \odot v)$.

$$\begin{aligned}
(-a) \odot v &= (a \cdot ___) \odot v \\
&= a \odot (-1 \odot v) \text{ by Axiom } ___ \text{ of a vector space} \\
&= a \odot (____) \text{ by part } ___
\end{aligned}$$

and

$$\begin{aligned}
(-a) \odot v &= (___ \cdot a) \odot v \\
&= -1 \odot (a \odot v) \text{ by Axiom } ___ \text{ of a vector space} \\
&= -(_____) \text{ by part } ___
\end{aligned}$$