Spring 2007
H. Jordon
100 points

Math 336 – Take-Home Exam 1

Instructions: You must do enough of the following problems to get exactly 100 points. Graduate students must do problems 3, 11, 14, and 16, and any of the remaining problems to reach 100 points. Each problem must be completed entirely, i.e., you may not select parts (a) and (c) only. Your resources for this exam are your class notes, our textbook (and only our textbook), and me. **You are not to discuss this exam with anyone else.** You are allowed to use any Theorem/Lemma/Corollary/Definition/Collected & Graded Exercise from Chapters 0–4 without proof. You may not cite Examples from the book; however, you may use examples from the book if you put them in your own writing. If your solution to one of the problems below depends on an exercise from the book that was not collected and graded or on another problem below, you must include a proof of that exercise and/or do that problem.

Write your solutions neatly, one to page, and submit your solutions **in order**, stapled in the upper left-hand corner. Your exam must include the following signed statement on the last page:

Signed Statement: With my signature below, I certify that I have only consulted my notes, our book, and Dr. Jordon in arriving at my answers on this exam.

Signed: ________________________

Due: 26 February, 12:00 PM

Throughout this exam, $\mathbb{Z}$ denotes the set of integers and $\mathbb{Z}_n$ denotes the integers modulo n.

Problems:

1. (5 pts.) Let G be a group. Prove that if $a^2b^2 = (ab)^2$ for all $a, b \in G$, then G is abelian.

2. (5 pts.) Prove that there is no group with exactly three elements of order 3.

3. (10 pts.) Let G be a group and let $a, b, \in G$. Show that $|ab| = |ba|$.

4. (5 pts.) Let $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid (ad - bc) \bmod 6 \neq 0 \right\}$, where $a, b, c, d \in \mathbb{Z}_6$ and all arithmetic is done using modulo 6. Is G a group under matrix multiplication? Why or why not?

5. (5 pts.) Suppose that a and b are group elements and $a^{-1}ba = b^{-1}$ and $b^{-1}ab = a^{-1}$. Prove that $a^2 = b^{-2}$.

6. (5 pts.) Give an example of a group that has subgroups of orders 1, 2, 3, 4, 5, and 6 but does not have a subgroup of order 7 or 8.

7. (5 pts.) Give an example of a non-abelian group with exactly four elements of order 10.

8. (10 pts.) It is easy to see that $U(25) = \langle 2 \rangle$. Find all the other generators of $U(25)$.

9. (10 pts.) Prove that for any integer $n > 1$, the group $U(n^2 + 2n)$ is not cyclic.

10. (10 pts.) Let p be prime and let n be a positive integer. How many subgroups does $\mathbb{Z}_{p^n}$ have (including the trivial subgroup and the group itself)? Be sure to justify your answer.

11. (10 pts.) Let p be prime and let n be a positive integer. How many generators does $\mathbb{Z}_{p^n}$ have?

12. (10 pts.) Let G be a group and consider $Z(G)$, the center of G.

    (a) If $ab \in Z(G)$, is it true that $a, b \in Z(G)$?

    (b) If $ab \in Z(G)$, prove that $ab = ba$.

13. (5 pts.) Let G be a group. Prove or disprove: If every proper subgroup of G is cyclic, then G is cyclic.

14. (10 pts.) Let G be an abelian group and let $H, K \leq G$. Define the set $HK = \{ab \mid a \in H, b \in K\}$. Prove that $HK \leq G$. Give an example to show that the set HK may not be subgroup of G if G is not abelian.

15. (10 pts.) Let G be an abelian group and let k be a fixed positive integer. Let $H = \{a \in G \mid |a| \text{ divides } k\}$. Prove that $H \leq G$.

16. (10 pts.) Let G be a cyclic group of order n.

    (a) If G has no proper subgroups, that is, the only subgroups of G are $\{e\}$ and G, then what can you say about n? Prove your claim.

    (b) If G has exactly two proper subgroups, what can you say about n? (In this case, G would have four subgroups total: $\{e\}$, G, H and K where $H \neq K$ and $\{e\} < H, K < G$.) Again, prove your claim.