Spring 2007
H. Jordon
100 points

Math 336 – Take-Home Exam 2

Instructions: You must do enough of the following problems to get exactly 100 points. Graduate students must do Problems 4, 5, and 10. Each problem must be completed entirely, i.e., you may not select parts (a) and (c) only. Your resources for this exam are your class notes, our textbook (and only our textbook), and me. **You are not to discuss this exam with anyone else.** Write your solutions neatly, one to page, and submit your solutions in order, stapled in the upper left-hand corner. Your exam must include the following signed statement on the last page:

Signed Statement: With my signature below, I certify that I have only consulted my notes, our book, and Dr. Jordon in arriving at my answers on this exam.

Signed: ________________________

Due: 2 April, 12:00 PM

Problems:

1. (5 pts.) The *exponent* of a group is the smallest positive integer n such that $x^n = e$ for all $x \in G$. Prove that every finite group has an exponent that divides $|G|$.

2. (10 pts.) A subgroup H of a group G is called *characteristic* if, for every automorphism ϕ of G, $\phi(H) = H$.

   (a) Prove that every subgroup of a cyclic group is characteristic.

   (b) Prove that the center, $Z(G)$, of a group G is characteristic.

3. (10 pts.) A hexagonal prism is a prism composed of two hexagonal bases and six rectangular (square) sides. Given the labeling of the hexagonal prism in class, determine the group G of rotations, written in disjoint cycle notation.

4. (15 pts.) Determine all the subgroups of $S_3 \oplus \mathbb{Z}_4$. You do not have to justify that you have found all the subgroups but beware–make sure that you have found **all** of them. (**Hint:** One approach would be to find all the subgroups of order 1, all subgroups of order 2, all subgroups of order 3, etc., for each divisor of $|S_3 \oplus \mathbb{Z}_4|$.) To describe a subgroup, remember that it is enough to list the generators, with the proper notation, i.e., $\langle((12), 2), (\epsilon, 2)\rangle = \{(\epsilon, 0), ((12), 2), (\epsilon, 2), ((12), 0)\}$.

5. (10 pts.) Let $H, K \leq G$ and let $HK = \{hk \mid h \in H, k \in K\}$ and $KH = \{kh \mid k \in K, h \in H\}$. Prove that HK is a group if and only if $HK = KH$.

6. (15 pts.) Give an example of each of the following or state that no such example exists. In each case, provide justification for your answers.

   (a) Four nonisomorphic groups of order 12.

   (b) A nonabelian group of order p where p is prime.

   (c) A finite group G and a positive integer k such that $k \mid |G|$ yet G has no subgroup of order k.

   (d) A subgroup H of $\mathbb{Z}$ with $[\mathbb{Z} : H]$ finite.

   (e) A subgroup H of $\mathbb{Z}$ with $[\mathbb{Z} : H]$ infinite.

7. (10 pts.) What is the order of the largest cyclic subgroup of $\mathrm{Aut}(\mathbb{Z}_{720})$? Justify your answer.

8. (15 pts.) Let $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where $p_1, p_2, \ldots, p_k$ are distinct primes and let $G \approx \mathbb{Z}_n$. Then the integers $p_1^{r_1}, p_2^{r_2}, \ldots, p_k^{r_k}$ are called the *elementary divisors* of G. For example, if $G \approx \mathbb{Z}_{30} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_6$, then we also have that $G \approx (\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_3)$. Thus, the elementary divisors of G are $2, 3, 5, 2^3, 2, 3$ or $2, 2, 2^3, 3, 3, 5$. Of course, we could rearrange the factors in the direct product above to obtain $G \approx \mathbb{Z}_{120} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2$. Since $2 \mid 6$ and $6 \mid 120$, we call $120, 6, 2$ the invariant factors of G and say that G is of type $(120, 6, 2)$. That is, if $G \approx \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_\ell}$ where $n_1, n_2, \ldots, n_\ell$ are integers, all at least 2, and $n_{i+1} \mid n_i$ for each $i = 1, 2, \ldots, k-1$, then the integers $n_1, n_2, \ldots, n_\ell$ are called the *invariant factors* of G and G is of *type* $(n_1, n_2, \ldots, n_\ell)$.

   Find all groups of order 1440 that can be written as an external direct product of cyclic groups. For each such group of order 1440, list the invariant factors and the elementary divisors. You should organize your work in a table with the following headings:

Group	Invariant Factors	Elementary Divisors

9. (10 pts.) Consider the three groups: D_{33}, $D_{11} \oplus \mathbb{Z}_3$, and $D_3 \oplus \mathbb{Z}_{11}$. Determine, with explanation, whether any two of these three groups are isomorphic.

10. (10 pts.) Let G be an infinite group and let H and K be infinite subgroups of G such that $K \leq H$ and $[G : H]$ and $[H : K]$ are both finite. Prove that $[G : K]$ must also be finite and that, in fact, $[G : K] = [G : H][H : K]$.

11. (10 pts.) Prove that $S_3 \oplus S_4$ is not isomorphic to S_6.