

## Enumeration Homework

Permutations can be viewed as directed graphs in which every component is a labeled directed cycle. (Here we allow a directed loop and directed 2-cycle.)

1. Determine explicitly the exponential generating function for permutations, say

$$P(x) = \sum_{n=1}^{\infty} P_n \frac{x^n}{n!}$$

2. Let  $C(x)$  denote the exponential series for “connected” permutations, that is, permutations that form a single cycle. Determine the explicit function for  $C(x)$ .
3. What is the identity relating  $C(x)$  and  $P(x)$ ? Verify the identity algebraically.
4. Let  $E(x)$  be the exponential series for connected even permutations, that is, permutations that form a single even directed cycle. Let  $Q(x)$  be the series for permutations in which every cycle is even. Determine  $E(x)$  and  $Q(x)$ .
5. Let  $F(x)$  be the exponential series for connected odd permutations, that is, permutations that form a single odd directed cycle. Let  $R(x)$  be the series for permutations in which every cycle is odd. Determine  $F(x)$  and  $R(x)$ .
6. Find a relation between  $P(x)$ ,  $Q(x)$ , and  $R(x)$ . Can you give a direct explanation for this relation?
7. Explicitly evaluate  $Q_n$  for all  $n \leq 10$ .
8. Examine the answers in Problem 7. You may wish to factor them! Can you conjecture a direct formula for  $Q_n$ ? Can you prove that this formula is correct?
9. Define  $I(x)$  to the exponential function for “identity” permutations. (Hint: There can be only one identity permutation of each order  $n$ .) Define  $D(x)$  to be the exponential generating function for “derangements”, that is, permutation without any fixed points. For example,  $\pi = (1234)(56)(78)$  is a derangement in  $S_8$ , the symmetric group on 8 elements while  $\rho = (1234)(678)$  is not a derangement since 5 is a fixed point. Use the multiplication principle to find an identity involving  $P(x)$ ,  $I(x)$ , and  $D(x)$ . Solve this identity for  $D(x)$ , and by evaluating the coefficient of  $x^n$ , find an explicit formula for  $D_n$ . (Note: You may have seen  $D_n$  solved by other methods in other courses, but please stick to the material at hand!) Verify that your formula is correct by finding the number of derangements using your formula for  $n = 1, 2, 3, 4$ .

10. H & P, pp. 29, 1.1. Verify that your formula is correct by finding the number of connected labeled digraphs of order  $p$  for  $p = 1, 2, 3$ .
11. H & P, pp. 29, 1.3. Verify that your formula is correct by finding the number of oriented labeled graphs (or signed labeled graphs) of order 3.
12. H & P, pp. 29, 1.4. Illustrate your formula for graphs of order at most 4.
13. H & P, pp. 29, 1.6, but note that the answer is actually  $2^{\lceil \frac{n-1}{2} \rceil}$ .
14. H & P, pp. 30, 1.7. This means multigraphs in which every vertex has even degree, and loops are permitted where each loop contributes 2 to its vertex's degree. The answer should be in the form of a generating function with the variable  $x$  representing edges. You may assume formula (1.4.7) and express your answer as  $W_p(x)$  times suitable factors.
15. (a) Compute  $W_6(x)$ . You may use a computer algebra system if you wish or do this by hand.  
 (b) For which  $p$  does  $W_p(x)$  have symmetric coefficients (that is, coefficients of  $x$  to the  $q$  and  $x$  to the  $\binom{p}{2} - q$  match) and for which does it not? There is a simple "proof" of symmetry in the cases that do have symmetry based on a graphical observation. There is also a purely algebraic "explanation".
16. It turns out that the "usual" method to count "connected colored labeled graphs" in terms of labeled colored graphs with exactly  $k$  "interchangeable" colors does not work. However, it is possible to obtain connected numbers by the following approach. Let's carry this out for  $k \leq 3$  and  $p \leq 5$ .
  - (a) Adjust the chart on page 18 (for  $k \leq 3$  and  $p \leq 5$ ) to make the colors non-interchangeable. This merely multiplies column  $k$  by  $k!$ .
  - (b) Now adjust the chart so the entries count labeled graphs colored with "at most  $k$ " colors. For example, the single coloring using  $k = 1$  color contributes 2 extra colorings in the "at most 2" column, and 3 extra in the "at most 3" column. Any 2-colored labeled graph contributes 3 to the "at most 3" column.
  - (c) Now count "connected, labeled, at most  $k$  colored graphs" by the usual technique.
  - (d) Now adjust the count to give connected colorings with exactly  $k$  colors appearing.
  - (e) Finally, adjust to give connected colorings with  $k$  interchangeable colors.
  - (f) Draw connected pictures to verify these numbers for  $k \leq 3$  and  $p \leq 5$ .
17. Count the number of labeled trees of order  $p$  with
  - (a) exactly two vertices of degree 1;
  - (b) exactly  $p - 2$  vertices of degree 1.

18. Use the Matrix Tree Theorem to find a simple formula for the number of spanning trees of  $K_{m,n}$ .
19. Let  $V(K_n) = \{v_1, v_2, \dots, v_n\}$ . Determine the number of spanning trees of  $K_n$  in which  $v_n$  is an end-vertex. (You might try  $n = 4$  first to get an idea.)
20. Find non-isomorphic permutations groups  $A$  and  $B$  with  $Z(A) = Z(B)$ .
21. Find isomorphic permutations groups  $A$  and  $B$  with  $Z(A) \neq Z(B)$ .
22. Find isomorphic, nonidentical permutation groups  $A$  and  $B$  on sets  $X$  and  $Y$  respectively with  $|X| = |Y|$  and  $Z(A) = Z(B)$ .
23. H & P, 2.5
24. H & P, 2.8