Skolem, Langford, Extended, and Near-Skolem Sequences, Oh My!

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DISC MATH
Skolem Sequences

Definition
A Skolem sequence $S$ of order $t$ is a sequence $S = s_1 s_2 ... s_{2t}$ of $2t$ integers such that

(S1) for every $\ell \in \{1, 2, ..., t\}$, there exists unique $s_i, s_j \in S$ such that $s_i = s_j = \ell$, and

(S2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$.

Example
A Skolem sequence of order $t = 5$:

$5242354311$
A **Skolem sequence** of order \( t \)

- can be written as a **collection of ordered pairs**

\[
\{(a_i, b_i) \mid 1 \leq i \leq t, \ b_i - a_i = i\}
\]

with

\[
\bigcup_{i=1}^{t} \{a_i, b_i\} = \{1,2,\ldots,2t\}
\]

- gives a **partition** of the set \( \{1, 2, \ldots, 3t\} \) into triples

\((a_i, b_i, c_i)\) such that \( a_i + b_i = c_i \) for each \( i = 1, 2, \ldots, t \).

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Ordered Pairs</th>
<th>Partition of ( {1, 2, \ldots, 15} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5242354311</td>
<td>{9, 10}</td>
<td>1 + 14 = 15</td>
</tr>
<tr>
<td>6789...</td>
<td>{2, 4}</td>
<td>2 + 7 = 9</td>
</tr>
<tr>
<td></td>
<td>{5, 8}</td>
<td>3 + 10 = 13</td>
</tr>
<tr>
<td></td>
<td>{3, 7}</td>
<td>4 + 8 = 12</td>
</tr>
<tr>
<td></td>
<td>{1, 6}</td>
<td>5 + 6 = 11</td>
</tr>
</tbody>
</table>
**Theorem** (Skolem, 1957) A Skolem sequence of order \( t \) exists if and only if \( t \equiv 0, 1 \pmod{4} \).

**Necessity**

A Skolem sequence of order \( t \)

\[ \Rightarrow \text{partition of } \{1, 2, \ldots, 3t\} \text{ into triples } (a_i, b_i, c_i) \text{ such that} \]

\[ a_i + b_i = c_i \text{ for each } i = 1, 2, \ldots, t \]

\[ a_i + b_i - c_i = 0 \]

\[ \Rightarrow \text{each triple must have an even number of odds} \]

\[ \Rightarrow \{1, 2, \ldots, 3t\} \text{ must have an even number of odds} \]

\[ \Rightarrow 3t \equiv 3, 4 \pmod{4} \Rightarrow t \equiv 0, 1 \pmod{4} \]
What about \( t \equiv 2, 3 \pmod{4} \)?

**Definition**
A hooked Skolem sequence \( HS \) of order \( t \) is a sequence \( HS = s_1 s_2 ... s_{2t+1} \) of \( 2t + 1 \) integers such that

(S1) for every \( l \in \{1, 2, ..., t\} \), there exists unique \( s_i, s_j \in HS \) such that \( s_i = s_j = l \),

(S2) if \( s_i = s_j = l \) with \( i < j \), then \( j - i = l \), and

(S3) \( s_{2t} = \_\_\_\_\_\_ \)

**Example**
A hooked Skolem sequence of order \( t = 6 \):

\[ 64511465232_3 \]
A **hooked Skolem sequence** of order \( t \)

- can be written as a **collection of ordered pairs** \( \{(a_i, b_i) \mid 1 \leq i \leq t, b_i - a_i = i\} \) with
  \[
  \bigcup_{i=1}^{t}\{a_i, b_i\} = \{1, 2, \ldots, 2t - 1, 2t + 1\}
  \]

- gives a **partition** of the set \( \{1, 2, \ldots, 3t - 1, 3t + 1\} \) into triples \((a_i, b_i, c_i)\) such that \( a_i + b_i = c_i \) for each \( i = 1, 2, \ldots, t \).

**Theorem** (O’Keefe, 1961) A hooked Skolem sequence of order \( t \) exists if and only if \( t \equiv 2, 3 \pmod{4} \).
Skolem sequences and their many generalizations have applications in numerous areas:

- triple systems, cyclically decomposing complete graphs into 3-cycles
- balanced ternary designs
- labelings of graphs, including labeling graphs to enhance testing the reliability of a communication network
- generating missile guidance codes resistant to random interference
- design of statistical models, such as a balanced sampling plan excluding contiguous units and a balanced sampling plan avoiding the selection of adjacent units
- Wythoff pairs
- construction of binary sequences with controllable complexity
- testing new parallel processing algorithms

A decomposition of a graph $G$ is a partition of its edge set into subsets.
**Triple systems** are decompositions of the complete graph \( K_n \) into 3-cycles.

\[ K_7 \text{ into 3-cycles.} \]
Necessary Conditions for Triple Systems

If a decomposition of $K_n$ into 3-cycles exists, then

- $3 \leq n$,
- $n$ is odd, and
- $3 \mid n(n - 1)/2$

$$3k = \frac{n(n-1)}{2} \quad \text{or} \quad 6 \mid n(n - 1) \quad \text{or} \quad n = 6t + 1, 6t + 3$$
Circulants

Let \( n > 1 \) and let \( L \subseteq \{1, 2, ..., \lfloor n/2 \rfloor \} \).

The **circulant graph** \( \langle L \rangle_n \) denotes that graph with vertex set \( \mathbb{Z}_n \) (the integers modulo \( n \)) and edge set
\[
\{ \{i, i + k\} \mid k \in L, i \in \mathbb{Z}_n \}.
\]

\[
\langle\{1, 3\}\rangle_{12}:
\]

\[
K_n = \langle\{1, 2, ..., \lfloor n/2 \rfloor\}\rangle_n
\]
Partition of \{1, 2, \ldots, 15\}

- 1 + 14 = 15
- 2 + 7 = 9
- 3 + 10 = 13
- 4 + 8 = 12
- 5 + 6 = 11

We have a decomposition of $K_{31}$ into 3-cycles!

$\langle\{1, 2, \ldots, 15\}\rangle_n$ into $5n$ 3-cycles
Existence of Triple Systems, 6t + 1 Case

**Corollary**  For \( t \geq 1 \), \( K_{6t+1} \) decomposes into 3-cycles.

**Proof**  Let \( t \geq 1 \).
Suppose first \( t \equiv 0, 1 \pmod{4} \). Then, there exists a Skolem sequence of order \( t \), giving a partition of \( \{1, 2, \ldots, 3t\} \) into \( t \) triples. These \( t \) triples give rise to a decomposition of \( K_{6t+1} = \langle\{1, 2, \ldots, 3t\}\rangle_{6t+1} \) into 3-cycles.

Now suppose \( t \equiv 2, 3 \pmod{4} \). Then, there exists a hooked Skolem sequence of order \( t \), giving a partition of \( \{1, 2, \ldots, 3t - 1, 3t + 1\} \) into \( t \) triples. Since
\[
\langle\{1, 2, \ldots, 3t\}\rangle_{6t+1} = \langle\{1, 2, \ldots, 3t - 1, 3t + 1\}\rangle_{6t+1},
\]
these \( t \) triples give rise to a decomposition of \( K_{6t+1} = \langle\{1, 2, \ldots, 3t\}\rangle_{6t+1} \) into 3-cycles.
Extended Skolem Sequences

**Definition**

A $k$-extended Skolem sequence $ES_k$ of order $t$ is a sequence $ES_k = s_1 \ s_2 \ ... \ s_{2t+1}$ of $2t + 1$ integers such that

(E1) for every $\ell \in \{1, 2, ..., t\}$, there exists unique $s_i, s_j \in ES_k$ such that $s_i = s_j = \ell$,

(E2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$, and

(E3) $s_k =$ __

**Example**

A $5$-extended Skolem sequence of order $t = 4$:

$$2423_{4311}$$

**Note:** A $(2t)$-extended Skolem sequence is a hooked Skolem sequence.
What kind of partitions do extended Skolem sequences provide?

A $k$-extended Skolem sequence of order $t$ provides a partition of $\{1, 2, \ldots, 3t + 1\} \setminus \{t + k\}$ into triples $(a_i, b_i, c_i)$ such that $a_i + b_i = c_i$ for $i = 1, 2, \ldots, t$.

**Example**

5-extended Skolem sequence of order $t = 4$:

$24234311$

$5678 \ldots$

$1 + 12 = 13$

$2 + 5 = 7$

$3 + 8 = 11$

$4 + 6 = 10$

$\{1, 2, \ldots, 13\} \setminus \{9\}$
5-extension Skolem, order 4

Partition of \( \{1, 2, \ldots, 13\} \setminus \{9\} \)

1 + 12 = 13
2 + 5 = 7
3 + 8 = 11
4 + 6 = 10

\( \langle \{1, 2, \ldots, 13\} \setminus \{9\}\rangle_n \)

into 4n 3-cycles

When \( n = 27 \),
we have \( K_{27} = \langle \{1, 2, \ldots, 13\}\rangle_{27} \) into 3-cycles.
**Hooked Extended Skolem Sequences**

**Definition**
A hooked \( k \)-extended Skolem sequence \( HES_k \) of order \( t \) is a sequence \( HES_k = s_1 s_2 \ldots s_{2t+2} \) of \( 2t + 2 \) integers such that

1. **(E1)** for every \( \ell \in \{1, 2, \ldots, t\} \), there exists unique \( s_i, s_j \in HES_k \) such that \( s_i = s_j = \ell \),
2. **(E2)** if \( s_i = s_j = \ell \) with \( i < j \), then \( j - i = \ell \), and
3. **(E3)** \( s_k = \_ \)
4. **(E4)** \( s_{2t+1} = \_ \)

**Example**
A hooked 4-extended Skolem sequence of order \( t = 4 \):

\[ 411_4232_3 \]
Hooked $k$-extended Skolem sequences

A hooked $k$-extended Skolem sequence of order $t$ provides a partition of $\{1, 2, \ldots, 3t + 2\} \setminus \{t + k, 3t + 1\}$ into triples $(a_i, b_i, c_i)$ such that $a_i + b_i = c_i$ for $i = 1, 2, \ldots, t$.

Example

hooked 4-extended Skolem sequence of order $t = 4$:

$$411_4232_3$$

1 + 6 = 7
2 + 10 = 12
3 + 11 = 14
4 + 5 = 9

$\{1, 2, \ldots, 14\} \setminus \{8, 13\}$
Theorem (Baker, 1995; Linek and Shalaby, 2008)

For positive integers $k$ and $t$ with $k \leq 2t + 1$, a $k$-extended Skolem sequence of order $t$ exists if and only if

- $k$ is odd and $t \equiv 0, 1 \pmod{4}$
- or

- $k$ is even and $t \equiv 2, 3 \pmod{4}$.

For positive integers $k$ and $t$ with $k < 2t + 1$, a hooked $k$-extended Skolem sequence of order $t$ exists if and only if

- $k$ is even and $t \equiv 0, 1 \pmod{4}$
- or

- $k$ is odd and $t \equiv 2, 3 \pmod{4}$.
Existence of Triple Systems, $6t + 3$ Case

**Corollary**  For $t \geq 1$, $K_{6t+3}$ decomposes into 3-cycles.

**Proof**  Let $t \geq 1$. Note $K_{6t+3} = \langle \{1, 2, \ldots, 3t + 1\}\rangle_{6t+3}$.  Suppose first $t \equiv 0, 3 \pmod{4}$. Then, there exists a $(t + 1)$-extended Skolem sequence of order $t$, giving a partition of $\{1, 2, \ldots, 3t + 1\}\backslash\{2t + 1\}$ into $t$ triples. These $t$ triples give rise to a decomposition of $\langle \{1, 2, \ldots, 3t + 1\}\backslash\{2t + 1\}\rangle_{6t+3}$ into 3-cycles. Since $\langle \{2t + 1\}\rangle_{6t+3}$ is a union of 3-cycles, we have a decomposition of $K_{6t+3}$ into 3-cycles.

Now suppose $t \equiv 1, 2 \pmod{4}$. Then, there exists a hooked $(t + 1)$-extended Skolem sequence of order $t$. We proceed as in the $t \equiv 0, 3 \pmod{4}$ case noting that $\langle \{1, 2, \ldots, 3t + 1\}\rangle_{6t+3} = \langle \{1, 2, \ldots, 3t, 3t + 2\}\rangle_{6t+3}$. 
Near Skolem Sequences

Definition
A near Skolem sequence $\text{NS}_k$ of order $t$ and defect $k$ is a sequence $\text{NS}_k = s_1 \ s_2 \ldots \ s_{2t-2}$ of $2t - 2$ integers such that

$(N1)$ for every $\ell \in \{1, 2, \ldots, t\}\setminus\{k\}$, there exists unique $s_i, s_j \in \text{NS}_k$ such that $s_i = s_j = \ell$,

$(N2)$ if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$, and

Example
A near Skolem sequence of order $t = 5$ and defect $k = 3$: $42524115$

Note: A near Skolem sequence of order $t$ and defect $t$ is a Skolem sequence of order $t - 1$. 
What kind of partitions do near Skolem sequences provide?

A near Skolem sequence of order $t$ and defect $k$ provides a partition of $\{1, 2, \ldots, 3t - 2\} \setminus \{k\}$ into triples $(a_i, b_i, c_i)$ such that $a_i + b_i = c_i$ for $i = 1, 2, \ldots, t$.

Example

near Skolem sequence of order $t = 5$ and defect $k = 3$:

$42524115$

$6 \ 7 \ 8 \ 9 \ \ldots$

$1 + 11 = 12$

$2 + 7 = 9$

$4 + 6 = 10$

$5 + 8 = 13$

$\{1, 2, \ldots, 13\} \setminus \{3\}$
Near Skolem, order 5, defect 3

Partition of \(\{1, 2, \ldots, 13\}\backslash\{3\}\)

\[
\begin{align*}
1 + 11 &= 12 \\
2 + 7 &= 9 \\
4 + 6 &= 10 \\
5 + 8 &= 13
\end{align*}
\]

\(\langle\{1, 2, \ldots, 13\}\backslash\{3\}\rangle_n\)

into \(4n\) 3-cycles

When \(n = 27\),

we have \(K_{27}\) into

108 3-cycles and

3 9-cycles.
Hooked Near Skolem Sequences

Definition
A hooked near Skolem sequence $HNS_k$ of order $t$ and defect $k$ is a sequence $HNS_k = s_1 s_2 ... s_{2t-1}$ of $2t - 1$ integers such that

(N1) for every $\ell \in \{1, 2, ..., t\}\backslash\{k\}$, there exists unique $s_i, s_j \in HNS_k$ such that $s_i = s_j = \ell$,

(N2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$, and

(N3) $s_{2t-2} =$ ___

Example
A hooked near Skolem sequence of order $t = 5$, defect $k = 2$: $4511435_3$
**Hooked Near Skolem sequences**

A hooked near Skolem sequence of order $t$ and defect $k$ provides a partition of $\{1, 2, \ldots, 3t - 1\} \setminus \{k, 3t - 2\}$ into triples $(a_i, b_i, c_i)$ such that $a_i + b_i = c_i$ for $i = 1, 2, \ldots, t$.

**Example**

hooked near Skolem sequence of order $t = 5$, defect $k = 2$:

$$4511435 \_3$$

$$1 + 8 = 9$$
$$3 + 11 = 14$$
$$4 + 6 = 10$$
$$5 + 7 = 12$$

$$\{1, 2, \ldots, 14\} \setminus \{2, 13\}$$
Let $k$ and $t$ be positive integers with $k \leq t$. A near Skolem sequence of order $t$ and defect $k$ exists if and only if

$k$ is odd and $t \equiv 0, 1 \pmod{4}$

or

$k$ is even and $t \equiv 2, 3 \pmod{4}$.

A hooked near Skolem sequence of order $t$ and defect $k$ exists if and only if

$k$ is even and $t \equiv 0, 1 \pmod{4}$

or

$k$ is odd and $t \equiv 2, 3 \pmod{4}$.
Langford Sequences

Definition
A Langford sequence \( L \) of order \( t \) and defect \( d \) is a sequence \( L = s_1 \ s_2 \ldots \ s_{2t} \) of \( 2t \) integers such that

\[ (L1) \quad \text{for every } \ell \in \{d, \ d + 1, \ d + 2, \ldots, \ d + t - 1\}, \text{ there exists unique } s_i, \ s_j \in L \text{ such that } s_i = s_j = \ell, \text{ and} \]

\[ (L2) \quad \text{if } s_i = s_j = \ell \text{ with } i < j, \text{ then } j - i = \ell. \]

Example
A Langford sequence of order \( t = 5 \) and defect \( d = 3 \):

\[ 7536435746 \]

Note: A Langford sequence of order \( t \) and defect \( d = 1 \) is a Skolem sequence of order \( t \).
Hooked Langford Sequences

**Definition**
A hooked Langford sequence $HL$ of order $t$ and defect $d$ is a sequence $HL = s_1 s_2 \ldots s_{2t+1}$ of $2t + 1$ integers such that

(L1) for every $\ell \in \{d, d + 1, d + 2, \ldots, d + t - 1\}$, there exists unique $s_i, s_j \in L$ such that $s_i = s_j = \ell$,
(L2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$, and
(L3) $s_{2t} = __$

**Example**
A hooked Langford sequence of order $t = 5$ and defect $d = 2$:

```
345364252_6
```
Partitions from (Hooked) Langford sequences

A (hooked) Langford sequence of order $t$ and defect $d$ provides a partition of $\{d, d + 1, d + 2, \ldots, d + 3t - 1\}$ into triples $(a_i, b_i, c_i)$ such that $a_i + b_i = c_i$ for $i = 1, 2, \ldots, t$.

**Example**

Langford sequence of order $t = 5$ and defect $d = 3$:

```
7536435746
8 9 10 ...
3 + 10 = 13
4 + 12 = 16
5 + 9 = 14
6 + 11 = 17
7 + 8 = 15
```

{3, 4, \ldots, 17}

Hooked Langford sequence of order $t = 5$ and defect $d = 2$:

```
345364252_6
2 + 13 = 15
3 + 7 = 10
4 + 8 = 12
5 + 9 = 14
6 + 11 = 17
```

{2, 3, \ldots, 15, 17}
Partition of $\{3, 4, ..., 17\}$

- $3 + 10 = 13$
- $4 + 12 = 16$
- $5 + 9 = 14$
- $6 + 11 = 17$
- $7 + 8 = 15$

$\langle\{3, 4, ..., 17\}\rangle_n$ into $5n$ $3$-cycles, $n \geq 35$
Theorem (Simpson, 1983)
A Langford sequence of order $t$ and defect $d$ exists if and only if
1. $t \geq 2d - 1$, and
2. $t \equiv 0, 1 \pmod{4}$ and $d$ is odd, or $t \equiv 0, 3 \pmod{4}$ and $d$ is even.

A hooked Langford sequence of order $t$ and defect $d$ exists if and only if
1. $t(t - 2d + 1) + 2 \geq 0$, and
2. $t \equiv 2, 3 \pmod{4}$ and $d$ is odd, or $t \equiv 1, 2 \pmod{4}$ and $d$ is even.
There are other interesting generalizations of Skolem sequences: \( k \)-extended, Langford and near-Skolem are just a few.

All of these ideas can be extended to \( m \)-tuples and integer partitioning.

**Definition**
An \( m \)-tuple \((d_1, d_2, \ldots, d_m)\) such that
\[
d_1 + d_2 + \ldots + d_m = 0
\]
is a **Skolem-type** \( m \)-tuple.

A set of \( t \) Skolem-type \( m \)-tuples whose entries, in absolute value, are \( \{1, 2, \ldots, mt\} \) is a **Skolem-type** \( m \)-tuple difference set of order \( t \).
**Examples of Skolem-type $m$-tuple difference sets**

**Skolem sequences** provide Skolem-type 3-tuple difference sets:

<table>
<thead>
<tr>
<th>Skolem sequence of order 5</th>
<th>Partition of ${1, 2, ..., 15}$</th>
<th>Skolem-type 3-tuple difference set of order 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>5242354311</td>
<td>$1 + 14 = 15$</td>
<td>$1 + 14 - 15 = 0$</td>
</tr>
<tr>
<td></td>
<td>$2 + 7 = 9$</td>
<td>$2 + 7 - 9 = 0$</td>
</tr>
<tr>
<td></td>
<td>$3 + 10 = 13$</td>
<td>$3 + 10 - 13 = 0$</td>
</tr>
<tr>
<td></td>
<td>$4 + 8 = 12$</td>
<td>$4 + 8 - 12 = 0$</td>
</tr>
<tr>
<td></td>
<td>$5 + 6 = 11$</td>
<td>$5 + 6 - 11 = 0$</td>
</tr>
</tbody>
</table>

**Skolem-type 5-tuple difference set of order 3**

<table>
<thead>
<tr>
<th></th>
<th>Partition of ${1, 2, ..., 15}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - 2 + 3 + 9 - 11 = 0$</td>
<td></td>
</tr>
<tr>
<td>$4 - 8 + 6 + 13 - 15 = 0$</td>
<td></td>
</tr>
<tr>
<td>$5 - 10 + 7 + 12 - 14 = 0$</td>
<td></td>
</tr>
</tbody>
</table>

**Note:** Every Skolem-type $m$-tuple has an even number of odds.
Example

\[ 4 - 8 + 6 + 13 - 15 = 0 \]

\( \langle \{4, 6, 8, 13, 15\} \rangle_n \) into \( n \) 5-cycles for all \( n \geq 31 \)
Existence of Skolem-type difference sets

**Theorem** (Bryant, J. & Ling, 2003)
- There exists a Skolem-type $m$-tuple difference set of order $t$ if and only if $mt \equiv 0, 3 \pmod{4}$.
- There exists a hooked Skolem-type $m$-tuple difference set of order $t$ if and only if $mt \equiv 1, 2 \pmod{4}$.

**Theorem** (Bryant, J. & Ling, 2003)
\[
\langle \{1, 2, \ldots, mt\} \rangle_n \text{ for } mt \equiv 0, 3 \pmod{4}
\]
and
\[
\langle \{1, 2, \ldots, mt-1, mt+1\} \rangle_n \text{ for } mt \equiv 1, 2 \pmod{4}
\]
decompose into $m$-cycles for all $n \geq 2mt + 1$ ($n \neq 2mt + 2$ when $mt \equiv 2 \pmod{4}$).
Extended Skolem-type Difference Sets

**Definition**
A \( k \)-extended \( m \)-tuple difference set of order \( t \) is a set of \( t \) Skolem-type \( m \)-tuples whose entries, in absolute value, are \( \{1, 2, ..., mt + 1\} \setminus \{k\} \).

**Example**
26-extended Skolem-type 5-tuple difference set of order 6: partition of \( \{1, 2, ..., 31\}\setminus\{26\} \) into 5-tuples

\[
\begin{align*}
4 + 15 - 17 + 18 - 20 &= 0 \\
5 + 9 - 12 + 21 - 23 &= 0 \\
6 + 10 - 14 + 22 - 24 &= 0 \\
7 + 11 - 16 + 25 - 27 &= 0 \\
8 + 13 - 19 + 28 - 30 &= 0 \\
1 + 3 - 2 + 29 - 31 &= 0 \\
\end{align*}
\]

**What can this give us?**
- decomposition of \( K_{63} \) into 5-cycles and one 63-cycle
- decomposition of \( K_{65} \) into 5-cycles, one 2-factor of 5-cycles, and one 65-cycle
- decomposition of \( K_n \) (\( n \geq 65 \)) into 5-cycles, one 2-factor with cycles of lengths \( n/\gcd(n, 26), \left\lfloor (n - 1)/2 \right\rfloor - 31 \) \( n \)-cycles, and a 1-factor if \( n \) is even
Existence of (hooked) extended Skolem-type 5-tuples

**Theorem** (Helms, J., Murray, Zeppetello, 2011)

- For positive integers \( k \) and \( t \) with \( k \leq 5t + 1 \), there exists a \( k \)-extended Skolem-type 5-tuple difference set of order \( t \) if and only if \( k \) is odd and \( t \equiv 0, 1 \pmod{4} \) or \( k \) is even and \( t \equiv 2, 3 \pmod{4} \).

- For positive integers \( k \) and \( t \) with \( k < 5t + 1 \), there exists a hooked \( k \)-extended Skolem-type 5-tuple difference set of order \( t \) if and only if \( k \) is odd and \( t \equiv 2, 3 \pmod{4} \) or \( k \) is even and \( t \equiv 0, 1 \pmod{4} \).
Corollary  (H., J., M., Z., 2011)
Let \( k, t, \) and \( n \) be positive integers with \( k \leq 5t + 1 \) and \( n \geq 10t + 3 \) with \( n \neq 10t + 4 \) when \( k \) is odd and \( t \equiv 2, 3 \mod 4 \) or \( k \) is even and \( t \equiv 0, 1 \mod 4 \).
Then \( K_n \) can be decomposed into
- \( tn \) 5-cycles,
- a 2-factor consisting of \( k \) cycles of length \( n/\gcd(n, k) \),
- \( \lceil (n - 1)/2 \rceil - (5t + 1) \) \( n \)-cycles, and
- a 1-factor if \( n \) is even.
Existence of Langford-type $m$-tuples

**Theorem** (Helms, J., Murray, Zeppetello, 2011)

There exists a Langford-type $m$-tuple difference set of order $t$ and defect $d$

- for all positive integers $t$ and $d$ when $m \equiv 0 \pmod{4}$;

- for all positive integers $t$ and $d$ with $t \equiv 0, 2 \pmod{4}$ when $m \equiv 2 \pmod{4}$;

- for all positive integers $t$ and $d$ with $2d - 1 \leq t$ and $t \equiv 0, 1 \pmod{4}$ if $d$ is odd, or $t \equiv 2, 3 \pmod{4}$ if $d$ is even when $m \equiv 3 \pmod{4}$;

- for all positive integers $t$ and $d$ with $t \equiv 0, 1 \pmod{4}$ and $2d \leq t$ if $d$ is even, or $t \equiv 0, 3 \pmod{4}$ and $2d \leq t - 5$ when $m \equiv 1 \pmod{4}$
Open Problems

Generalize any generalization of Skolem sequences to Skolem-type $m$-tuples!

Guiding Principle: Always partition a set with an even number of odd integers.
Thank you!