Skolem, Langford, Extended, and Near-Skolem Sequences, Oh My!

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3 November 2011
DISC MATH

Skolem Sequences

Definition

A Skolem sequence S of order t is a sequence $S = s_1 s_2 \dots s_{2t}$ of $2t$ integers such that

(S1) for every $\ell \in \{1, 2, \dots, t\}$, there exists unique $s_i, s_j \in S$ such that $s_i = s_j = \ell$, and

(S2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$.

Example

A Skolem sequence of order $t = 5$:

5242354311

A Skolem sequence of order t

- can be written as a collection of ordered pairs $\{(a_i, b_i) \mid 1 \le i \le t,\ b_i - a_i = i\}$ with $\bigcup_{i=1}^{t}\{a_i, b_i\} = \{1,2,\ldots,2t\}$

- gives a partition of the set $\{1, 2, \ldots, 3t\}$ into triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for each $i = 1, 2, \ldots, t$.

Sequence	Ordered Pairs	Partition of {1, 2, ..., 15}
5242354311 6 7 8 9 ...	{9, 10}	1 + 14 = 15
	{2, 4}	2 + 7 = 9
	{5, 8}	3 + 10 = 13
	{3, 7}	4 + 8 = 12
	{1, 6}	5 + 6 = 11

Theorem (Skolem, 1957) A Skolem sequence of order t exists if and only if $t \equiv 0, 1 \pmod 4$.

Necessity

A Skolem sequence of order t

$\Rightarrow$ partition of $\{1, 2, \dots, 3t\}$ into triples (a_i, b_i, c_i) such that

$$a_i + b_i = c_i \text{ for each } i = 1, 2, \dots, t$$

$$a_i + b_i - c_i = 0$$

$\Rightarrow$ each triple must have an even number of odds

$\Rightarrow$ $\{1, 2, \dots, 3t\}$ must have an even number of odds

$\Rightarrow$ $3t \equiv 3, 4 \pmod 4$ $\Rightarrow$ $t \equiv 0, 1 \pmod 4$

What about $t \equiv 2, 3 \pmod 4$?

Definition

A hooked Skolem sequence HS of order t is a sequence $HS = s_1 s_2 \ldots s_{2t+1}$ of $2t + 1$ integers such that

(S1) for every $\ell \in \{1, 2, \ldots, t\}$, there exists unique $s_i, s_j \in HS$ such that $s_i = s_j = \ell$,

(S2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$, and

(S3) $s_{2t} =$ ___

Example

A hooked Skolem sequence of order $t = 6$:

64511465232_3

A hooked Skolem sequence of order t

- can be written as a collection of ordered pairs $\{(a_i, b_i) \mid 1 \le i \le t,\ b_i - a_i = i\}$ with

$$\bigcup_{i=1}^{t} \{a_i, b_i\} = \{1, 2, \ldots, 2t-1, 2t+1\}$$

- gives a partition of the set $\{1, 2, \ldots, 3t-1, 3t+1\}$ into triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for each $i = 1, 2, \ldots, t$.

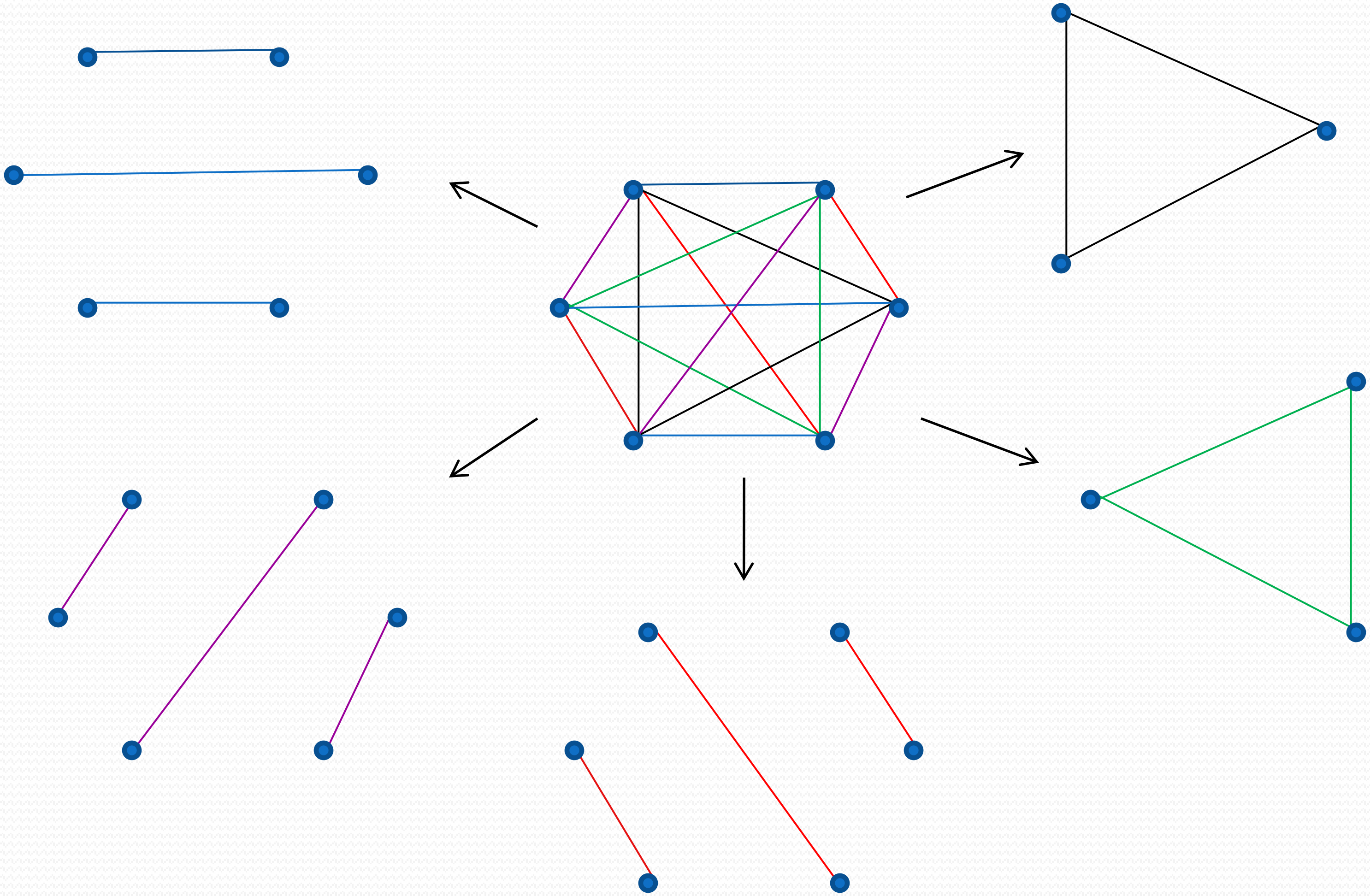
Theorem (O'Keefe, 1961) A hooked Skolem sequence of order t exists if and only if $t \equiv 2, 3 \pmod 4$.

Skolem sequences and their many generalizations have applications in numerous areas:

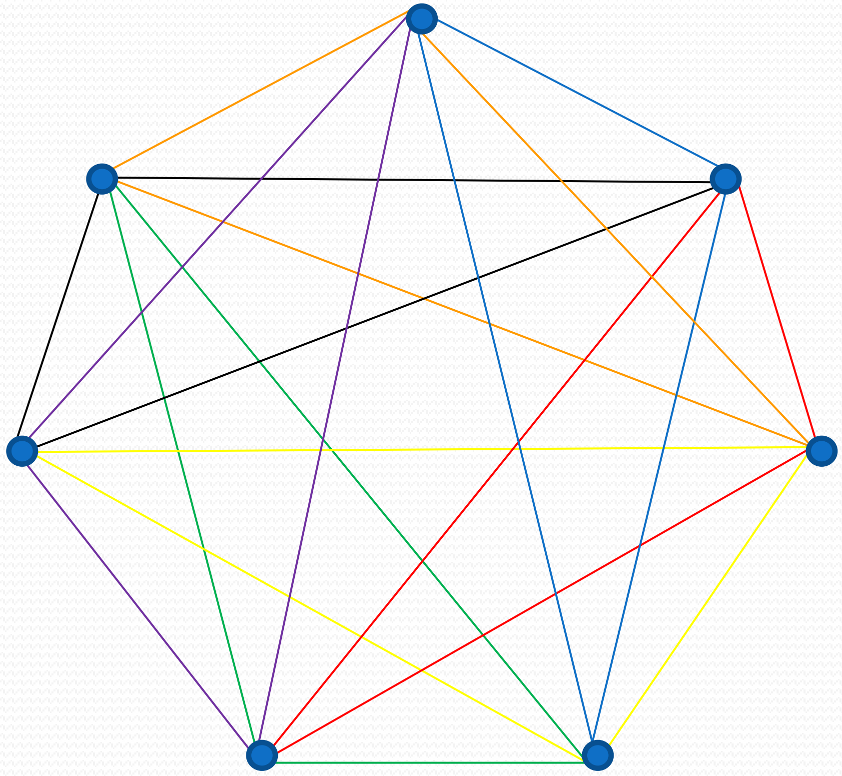
- triple systems, cyclically decomposing complete graphs into 3-cycles
- balanced ternary designs
- labelings of graphs, including labeling graphs to enhance testing the reliability of a communication network
- generating missile guidance codes resistant to random interference
- design of statistical models, such as a balanced sampling plan excluding contiguous units and a balanced sampling plan avoiding the selection of adjacent units
- Wythoff pairs
- construction of binary sequences with controllable complexity
- testing new parallel processing algorithms

See Nevena Francetić, and Eric Mendelsohn, A Survey of Skolem-type sequences and Rosa's use of them, *Mathematica Slovaca* **59** (2009) 39–76.

A decomposition of a graph G is a partition of its edge set into subsets.



Triple systems are decompositions of the complete graph K_n into 3-cycles.



K_7 into 3-cycles.

Necessary Conditions for Triple Systems

If a decomposition of K_n into 3-cycles exists, then

- $3 \leq n$,
- n is odd, and
- $3 \mid n(n-1)/2$

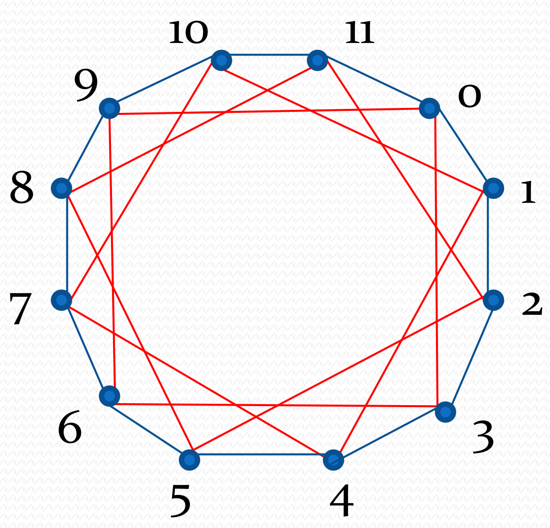
$$3k = \frac{n(n-1)}{2} \quad \text{or} \quad 6 \mid n(n-1) \quad \text{or} \quad n = 6t+1, 6t+3$$

Circulants

Let $n > 1$ and let $L \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$.

The circulant graph $\langle L \rangle_n$ denotes that graph with vertex set $\mathbb{Z}_n$ (the integers modulo n) and edge set $\{ \{i, i+k\} \mid k \in L, i \in \mathbb{Z}_n \}$.

$\langle\{1, 3\}\rangle_{12}$:



$K_n = \langle\{1, 2, \ldots, \lfloor n/2 \rfloor\}\rangle_n$

5242354311

Partition of {1, 2, …, 15}

1 + 14 = 15

2 + 7 = 9

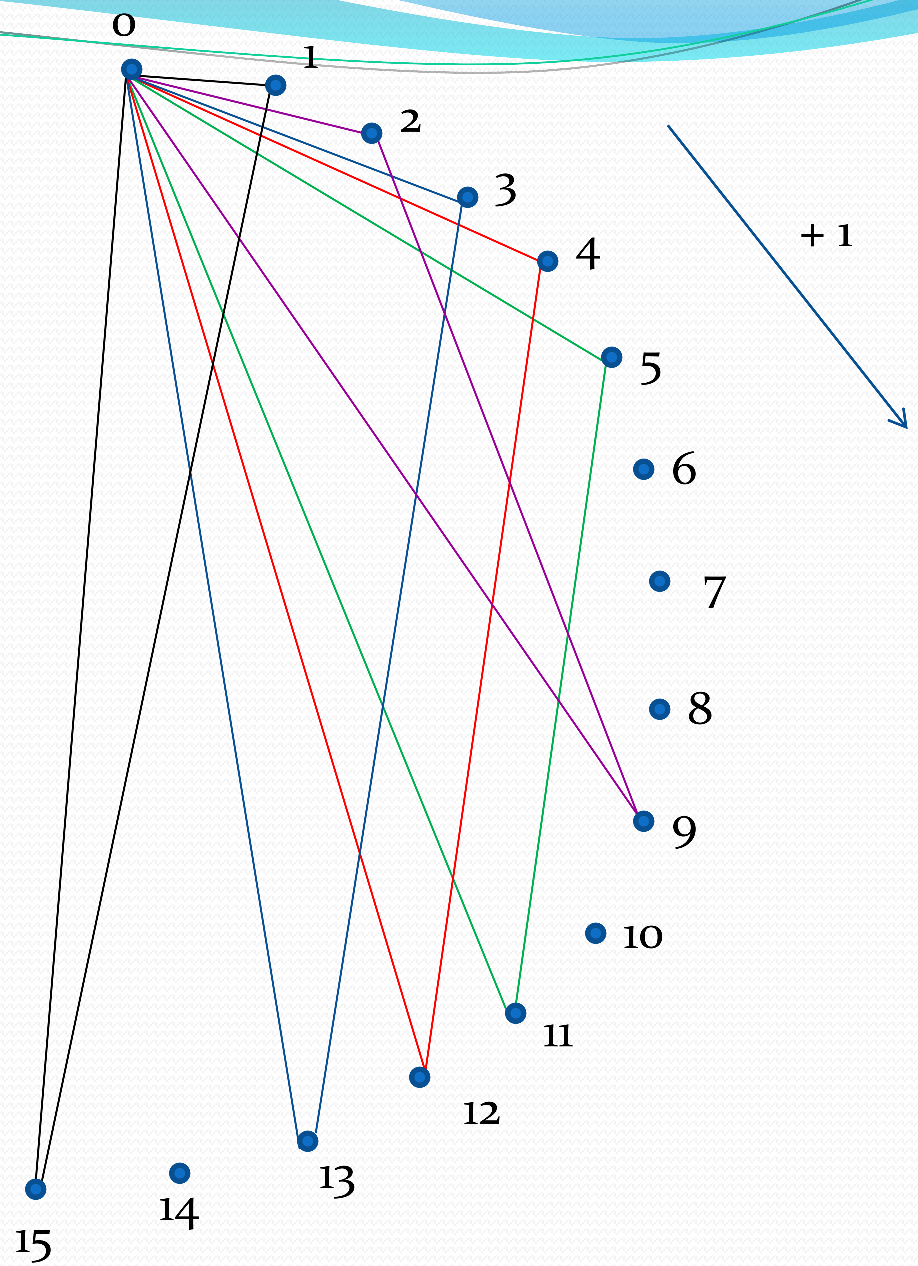
3 + 10 = 13

4 + 8 = 12

5 + 6 = 11

We have a decomposition of K_{31} into 3-cycles!

$\langle\{1, 2, \ldots, 15\}\rangle_n$ into $5n$ 3-cycles



Existence of Triple Systems, 6*t* + 1 Case

Corollary For $t \geq 1$, K_{6t+1} decomposes into 3-cycles.

Proof Let $t \geq 1$.
Suppose first $t \equiv 0, 1 \pmod 4$. Then, there exists a Skolem sequence of order t, giving a partition of $\{1, 2, \ldots, 3t\}$ into t triples. These t triples give rise to a decomposition of $K_{6t+1} = \langle\{1, 2, \ldots, 3t\}\rangle_{6t+1}$ into 3-cycles.

Now suppose $t \equiv 2, 3 \pmod 4$. Then, there exists a hooked Skolem sequence of order t, giving a partition of $\{1, 2, \ldots, 3t - 1, 3t + 1\}$ into t triples. Since

$$\langle\{1, 2, \ldots, 3t\}\rangle_{6t+1} = \langle\{1, 2, \ldots, 3t-1, 3t+1\}\rangle_{6t+1},$$

these t triples give rise to a decomposition of $K_{6t+1} = \langle\{1, 2, \ldots, 3t\}\rangle_{6t+1}$ into 3-cycles.

Extended Skolem Sequences

Definition

A k-extended Skolem sequence ES_k of order t is a sequence $ES_k = s_1 s_2 \ldots s_{2t+1}$ of $2t + 1$ integers such that

(E1) for every $\ell \in \{1, 2, \ldots, t\}$, there exists unique $s_i, s_j \in ES_k$ such that $s_i = s_j = \ell$,

(E2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$, and

(E3) $s_k =$ __

Example

A 5-extended Skolem sequence of order $t = 4$:

2423_4311

Note: A $(2t)$-extended Skolem sequence is a hooked Skolem sequence.

What kind of partitions do extended Skolem sequences provide?

A k-extended Skolem sequence of order t provides a partition of $\{1, 2, \ldots, 3t + 1\} \setminus \{t + k\}$ into triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for $i = 1, 2, \ldots, t$.

Example

5-extended Skolem sequence of order $t = 4$:

2423_4311
5 6 7 8 …

$1 + 12 = 13$

$2 + 5 = 7$

$3 + 8 = 11$

$4 + 6 = 10$

$\{1, 2, \ldots, 13\} \setminus \{9\}$

5-extended Skolem, order 4

Partition of {1, 2, …, 13}\{9}

1 + 12 = 13

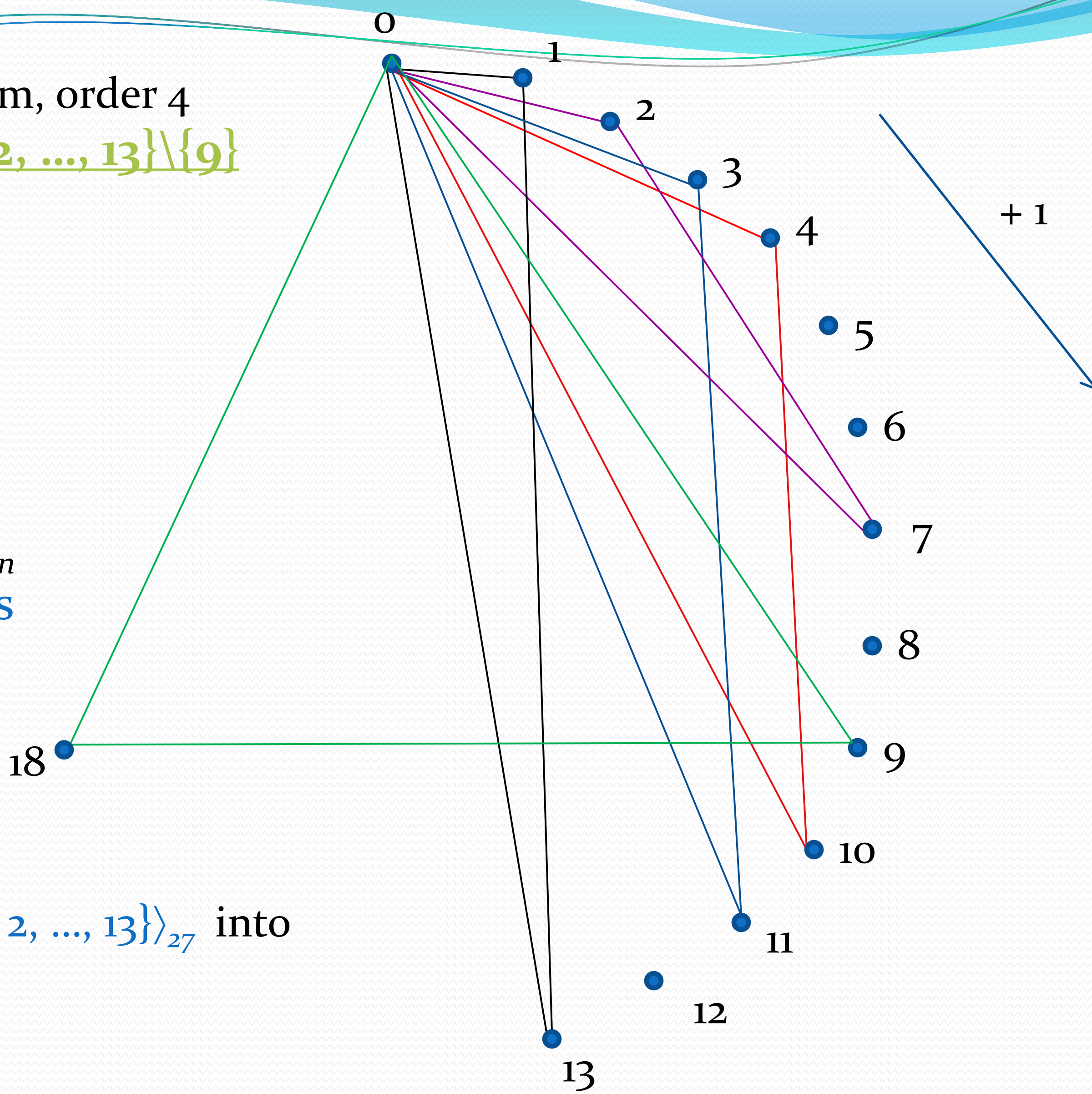
2 + 5 = 7

3 + 8 = 11

4 + 6 = 10

$\langle\{1, 2, \dots, 13\}\backslash\{9\}\rangle_n$
into $4n$ 3-cycles

When $n = 27$,
we have $K_{27} = \langle\{1, 2, \dots, 13\}\rangle_{27}$ into 3-cycles.



Hooked Extended Skolem Sequences

Definition

A hooked k-extended Skolem sequence HES_k of order t is a sequence $HES_k = s_1\, s_2 \ldots s_{2t+2}$ of $2t + 2$ integers such that

(E1) for every $\ell \in \{1, 2, \ldots, t\}$, there exists unique $s_i, s_j \in HES_k$ such that $s_i = s_j = \ell$,

(E2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$, and

(E3) $s_k = \_\_$

(E4) $s_{2t+1} = \_\_$

Example

A hooked 4-extended Skolem sequence of order $t = 4$:

411_4232_3

Hooked *k*-extended Skolem sequences

A hooked k-extended Skolem sequence of order t provides a partition of $\{1, 2, \ldots, 3t + 2\} \setminus \{t + k, 3t + 1\}$ into triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for $i = 1, 2, \ldots, t$.

Example

hooked 4-extended Skolem sequence of order $t = 4$:

411_4232_3

$1 + 6 = 7$

$2 + 10 = 12$

$3 + 11 = 14$

$4 + 5 = 9$

$\{1, 2, \ldots, 14\} \setminus \{8, 13\}$

Theorem (Baker, 1995; Linek and Shalaby, 2008)

For positive integers k and t with $k \le 2t+1$, a k-extended Skolem sequence of order t exists if and only if

k is odd and $t \equiv 0, 1 \pmod 4$

or

k is even and $t \equiv 2, 3 \pmod 4$.

For positive integers k and t with $k < 2t+1$, a hooked k-extended Skolem sequence of order t exists if and only if

k is even and $t \equiv 0, 1 \pmod 4$

or

k is odd and $t \equiv 2, 3 \pmod 4$.

Existence of Triple Systems, $6t + 3$ Case

Corollary For $t \geq 1$, K_{6t+3} decomposes into 3-cycles.

Proof Let $t \geq 1$. Note $K_{6t+3} = \langle\{1, 2, \dots, 3t + 1\}\rangle_{6t+3}$. Suppose first $t \equiv 0, 3 \pmod 4$. Then, there exists a $(t + 1)$-extended Skolem sequence of order t, giving a partition of $\{1, 2, \dots, 3t + 1\}\backslash\{2t + 1\}$ into t triples. These t triples give rise to a decomposition of $\langle\{1, 2, \dots, 3t + 1\}\backslash\{2t + 1\}\rangle_{6t+3}$ into 3-cycles. Since $\langle\{2t + 1\}\rangle_{6t+3}$ is a union of 3-cycles, we have a decomposition of K_{6t+3} into 3-cycles.

Now suppose $t \equiv 1, 2 \pmod 4$. Then, there exists a hooked $(t + 1)$-extended Skolem sequence of order t. We proceed as in the $t \equiv 0, 3 \pmod 4$ case noting that $\langle\{1, 2, \dots, 3t + 1\}\rangle_{6t+3}$ $= \langle\{1, 2, \dots, 3t, 3t + 2\rangle_{6t+3}$.

Near Skolem Sequences

Definition

A near Skolem sequence NS_k of order t and defect k is a sequence $NS_k = s_1\, s_2\, \ldots\, s_{2t-2}$ of $2t-2$ integers such that

(N1) for every $\ell \in \{1, 2, \ldots, t\}\backslash\{k\}$, there exists unique $s_i, s_j \in NS_k$ such that $s_i = s_j = \ell$,

(N2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$, and

Example

A near Skolem sequence of order $t = 5$ and defect $k = 3$:

42524115

Note: A near Skolem sequence of order t and defect t is a Skolem sequence of order $t - 1$.

What kind of partitions do near Skolem sequences provide?

A near Skolem sequence of order t and defect k provides a partition of $\{1, 2, \ldots, 3t-2\} \setminus \{k\}$ into triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for $i = 1, 2, \ldots, t$.

Example

near Skolem sequence of order $t = 5$ and defect $k = 3$:

42524115

6 7 8 9 …

$1 + 11 = 12$

$2 + 7 = 9$

$4 + 6 = 10$

$5 + 8 = 13$

$\{1, 2, \ldots, 13\} \setminus \{3\}$

Near Skolem, order 5, defect 3

Partition of {1, 2, ..., 13}\{3}

1 + 11 = 12

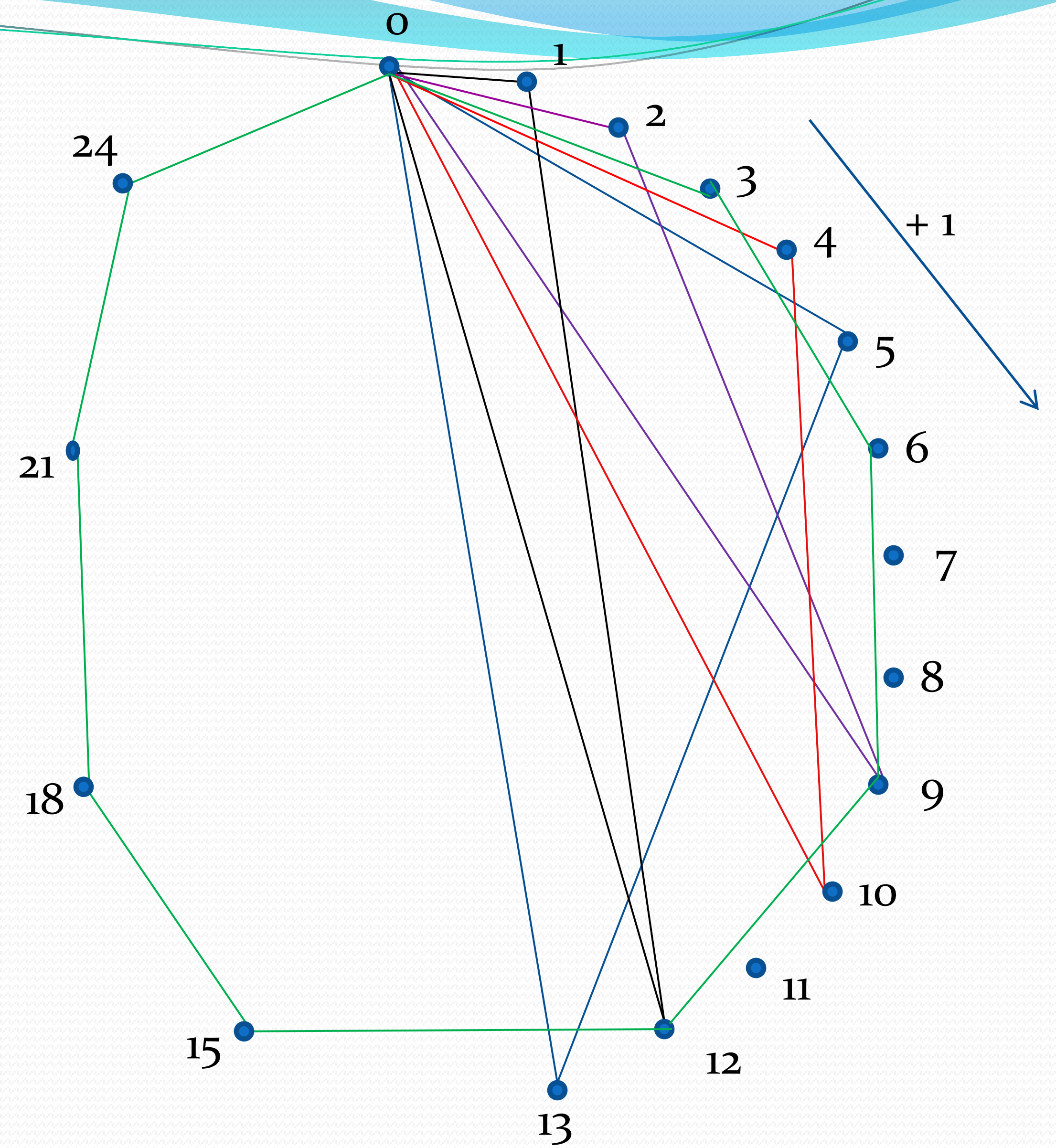
2 + 7 = 9

4 + 6 = 10

5 + 8 = 13

$\langle \{1, 2, \ldots, 13\} \setminus \{3\} \rangle_n$
into $4n$ 3-cycles

When $n = 27$,
we have K_{27} into
108 3-cycles and
3 9-cycles.



Hooked Near Skolem Sequences

Definition

A hooked near Skolem sequence HNS_k of order t and defect k is a sequence $HNS_k = s_1\, s_2 \ldots s_{2t-1}$ of $2t - 1$ integers such that

(N1) for every $\ell \in \{1, 2, \ldots, t\}\backslash\{k\}$, there exists unique $s_i, s_j \in HNS_k$ such that $s_i = s_j = \ell$,

(N2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$, and

(N3) $s_{2t-2} = \_\_$

Example

A hooked near Skolem sequence of order $t = 5$, defect $k = 2$:

4511435_3

Hooked Near Skolem sequences

A hooked near Skolem sequence of order t and defect k provides a partition of $\{1, 2, \ldots, 3t-1\} \setminus \{k, 3t-2\}$ into triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for $i = 1, 2, \ldots, t$.

Example

hooked near Skolem sequence of order $t = 5$, defect $k = 2$:

4511435_3

$1 + 8 = 9$

$3 + 11 = 14$

$4 + 6 = 10$

$5 + 7 = 12$

$\{1, 2, \ldots, 14\} \setminus \{2, 13\}$

Theorem (Shalaby, 1994)

Let k and t be positive integers with $k \le t$.

A near Skolem sequence of order t and defect k exists if and only if

k is odd and $t \equiv 0, 1 \pmod 4$

or

k is even and $t \equiv 2, 3 \pmod 4$.

A hooked near Skolem sequence of order t and defect k exists if and only if

k is even and $t \equiv 0, 1 \pmod 4$

or

k is odd and $t \equiv 2, 3 \pmod 4$.

Langford Sequences

Definition

A Langford sequence L of order t and defect d is a sequence $L = s_1\, s_2 \ldots s_{2t}$ of $2t$ integers such that

(L1) for every $\ell \in \{d, d+1, d+2, \ldots, d+t-1\}$, there exists unique $s_i, s_j \in L$ such that $s_i = s_j = \ell$, and

(L2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$.

Example

A Langford sequence of order $t = 5$ and defect $d = 3$:

7536435746

Note: A Langford sequence of order t and defect $d = 1$ is a Skolem sequence of order t.

Hooked Langford Sequences

Definition

A hooked Langford sequence HL of order t and defect d is a sequence $HL = s_1\, s_2 \ldots s_{2t+1}$ of $2t + 1$ integers such that

(L1) for every $\ell \in \{d, d+1, d+2, \ldots, d+t-1\}$, there exists unique $s_i, s_j \in L$ such that $s_i = s_j = \ell$,

(L2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$, and

(L3) $s_{2t} =$ __

Example

A hooked Langford sequence of order $t = 5$ and defect $d = 2$:

345364252_6

Partitions from (Hooked) Langford sequences

A (hooked) Langford sequence of order t and defect d provides a partition of $\{d, d+1, d+2, \ldots, d+3t-1\}$ ($\{d, d+1, d+2, \ldots, d+3t-2, d+3t\}$) into triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for $i = 1, 2, \ldots, t$.

Example

Langford sequence of order $t = 5$ and defect $d = 3$:

7536435746
8 9 10 …

$3 + 10 = 13$
$4 + 12 = 16$
$5 + 9 = 14$
$6 + 11 = 17$
$7 + 8 = 15$

$\{3, 4, \ldots, 17\}$

Hooked Langford sequence of order $t = 5$ and defect $d = 2$:

345364252_6

$2 + 13 = 15$
$3 + 7 = 10$
$4 + 8 = 12$
$5 + 9 = 14$
$6 + 11 = 17$

$\{2, 3, \ldots, 15, 17\}$

Partition of {3, 4, ..., 17}

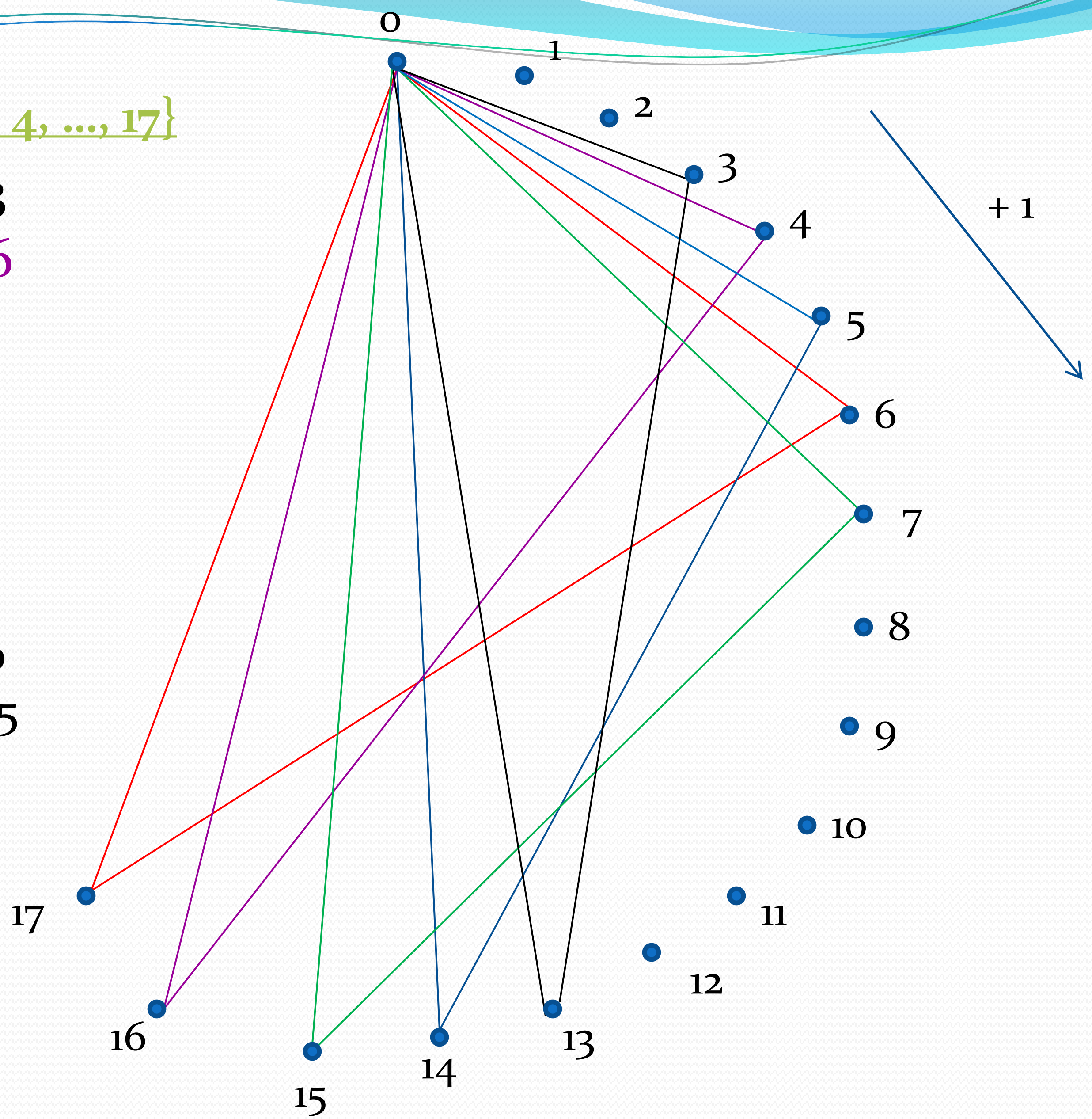
3 + 10 = 13

4 + 12 = 16

5 + 9 = 14

6 + 11 = 17

7 + 8 = 15



$\langle\{3, 4, \ldots, 17\}\rangle_n$ into $5n$ 3-cycles, $n \geq 35$

Theorem (Simpson, 1983)

A Langford sequence of order t and defect d exists if and only if

1. $t \geq 2d - 1$, and
2. $t \equiv 0, 1 \pmod{4}$ and d is odd, or $t \equiv 0, 3 \pmod{4}$ and d is even.

A hooked Langford sequence of order t and defect d exists if and only if

1. $t(t - 2d + 1) + 2 \geq 0$, and
2. $t \equiv 2, 3 \pmod{4}$ and d is odd, or $t \equiv 1, 2 \pmod{4}$ and d is even.

There are other interesting generalizations of Skolem sequences: *k*-extended, Langford and near-Skolem are just a few.

All of these ideas can be extended to *m*-tuples and integer partitioning.

Definition

An *m*-tuple $(d_1, d_2, \dots, d_m)$ such that

$$d_1 + d_2 + \dots + d_m = 0$$

is a Skolem-type *m*-tuple.

A set of t Skolem-type *m*-tuples whose entries, in absolute value, are $\{1, 2, \dots, mt\}$ is a Skolem-type *m*-tuple difference set of order t.

Examples of Skolem-type *m*-tuple difference sets

Skolem sequences provide Skolem-type 3-tuple difference sets:

Skolem sequence of order 5	Partition of {1, 2, …, 15}	Skolem-type 3-tuple difference set of order 5
5242354311	$1 + 14 = 15$	$1 + 14 - 15 = 0$
	$2 + 7 = 9$	$2 + 7 - 9 = 0$
	$3 + 10 = 13$	$3 + 10 - 13 = 0$
	$4 + 8 = 12$	$4 + 8 - 12 = 0$
	$5 + 6 = 11$	$5 + 6 - 11 = 0$

Skolem-type 5-tuple difference set of order 3

$1 - 2 + 3 + 9 - 11 = 0$

$4 - 8 + 6 + 13 - 15 = 0$

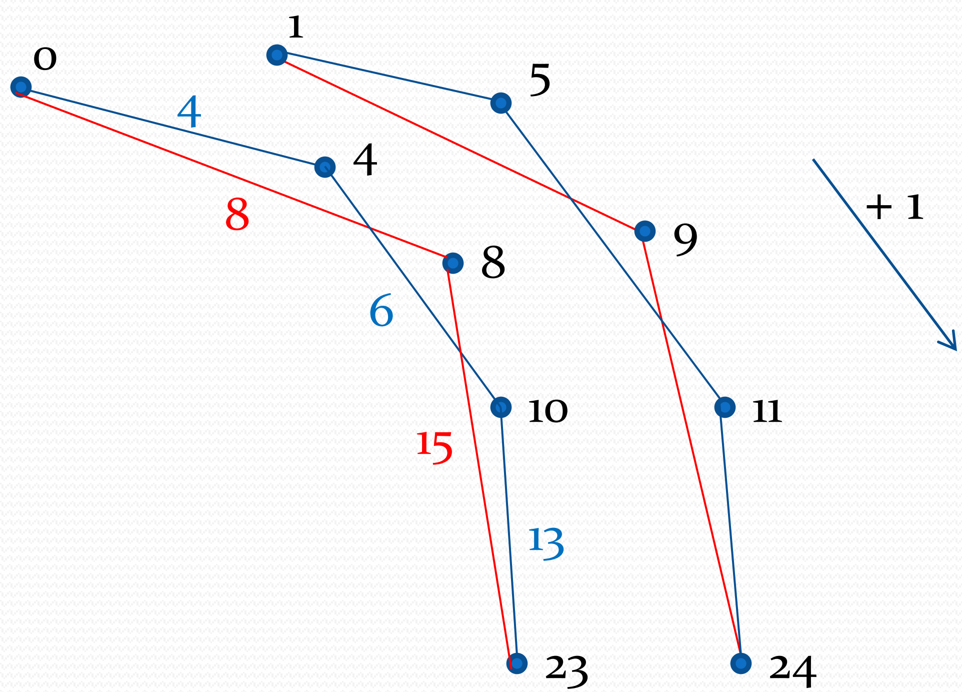
$5 - 10 + 7 + 12 - 14 = 0$

partition of {1, 2, … , 15}

Note: Every Skolem-type *m*-tuple has an even number of odds.

Example

$$4 - 8 + 6 + 13 - 15 = 0$$



$\langle\{4, 6, 8, 13, 15\}\rangle_n$ into n 5-cycles for all $n \geq 31$

Existence of Skolem-type difference sets

Theorem (Bryant, J. & Ling, 2003)

- There exists a Skolem-type m-tuple difference set of order t if and only if $mt \equiv 0, 3 \pmod 4$.
- There exists a hooked Skolem-type m-tuple difference set of order t if and only if $mt \equiv 1, 2 \pmod 4$.

Theorem (Bryant, J. & Ling, 2003)

$$\langle\{1, 2, \ldots, mt\}\rangle_n \text{ for } mt \equiv 0, 3 \pmod 4$$

and

$$\langle\{1, 2, \ldots, mt-1, mt+1\}\rangle_n \text{ for } mt \equiv 1, 2 \pmod 4$$

decompose into m-cycles for all $n \geq 2mt+1$ ($n \neq 2mt+2$ when $mt \equiv 2 \pmod 4$).

Extended Skolem-type Difference Sets

Definition

A k-extended Skolem-type m-tuple difference set of order t is a set of t Skolem-type m-tuples whose entries, in absolute value, are

$$\{1, 2, \ldots, mt + 1\} \setminus \{k\}.$$

Example

26-extended Skolem-type 5-tuple difference set of order 6:

partition of $\{1, 2, \ldots, 31\}\setminus\{26\}$ into 5-tuples

$4 + 15 - 17 + 18 - 20 = 0$
$5 + 9 - 12 + 21 - 23 = 0$
$6 + 10 - 14 + 22 - 24 = 0$
$7 + 11 - 16 + 25 - 27 = 0$
$8 + 13 - 19 + 28 - 30 = 0$
$1 + 3 - 2 + 29 - 31 = 0$

What can this give us?

- decomposition of K_{63} into 5-cycles and one 63-cycle
- decomposition of K_{65} into 5-cycles, one 2-factor of 5-cycles, and one 65-cycle
- decomposition of K_n $(n \geq 65)$ into 5-cycles, one 2-factor with cycles of lengths $n/\gcd(n, 26)$, $\lfloor (n-1)/2 \rfloor - 31$ n-cycles, and a 1-factor if n is even

Existence of (hooked) extended Skolem-type 5-tuples

Theorem (Helms, J., Murray, Zeppetello, 2011)

- For positive integers k and t with $k \leq 5t + 1$, there exists a k-extended Skolem-type 5-tuple difference set of order t if and only if k is odd and $t \equiv 0, 1 \pmod 4$ or
  k is even and $t \equiv 2, 3 \pmod 4$.

- For positive integers k and t with $k < 5t + 1$, there exists a hooked k-extended Skolem-type 5-tuple difference set of order t if and only if k is odd and $t \equiv 2, 3 \pmod 4$ or
  k is even and $t \equiv 0, 1 \pmod 4$.

And, the decompositions obtained are …

Corollary (H., J., M., Z., 2011)

Let k, t, and n be positive integers with $k \leq 5t + 1$ and $n \geq 10t + 3$ with $n \neq 10t + 4$ when k is odd and $t \equiv 2, 3 \pmod 4$ or k is even and $t \equiv 0, 1 \pmod 4$.

Then K_n can be decomposed into

- tn 5-cycles,
- a 2-factor consisting of k cycles of length $n/\gcd(n, k)$,
- $\lfloor (n - 1)/2 \rfloor - (5t + 1)$ n-cycles, and
- a 1-factor if n is even.

Existence of Langford-type *m*-tuples

Theorem (Helms, J., Murray, Zeppetello, 2011)

There exists a Langford-type m-tuple difference set of order t and defect d

- for all positive integers t and d when $m \equiv 0 \pmod 4$;
- for all positive integers t and d with $t \equiv 0, 2 \pmod 4$ when $m \equiv 2 \pmod 4$;
- for all positive integers t and d with $2d - 1 \le t$ and $t \equiv 0, 1 \pmod 4$ if d is odd, or $t \equiv 2, 3 \pmod 4$ if d is even when $m \equiv 3 \pmod 4$;
- for all positive integers t and d with $t \equiv 0, 1 \pmod 4$ and $2d \le t$ if d is even, or $t \equiv 0, 3 \pmod 4$ and $2d \le t - 5$ when $m \equiv 1 \pmod 4$

Open Problems

Generalize any generalization of Skolem sequences to Skolem-type m-tuples!

Guiding Principle: Always partition a set with an even number of odd integers.

Thank you!