Section 2.8: One-to-one Linear Transformations

Today we are going to learn about one-to-one linear transformations.

**Definition:** A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one if $u \neq v$ in $\mathbb{R}^n$, then $T(u) \neq T(v)$ in $\mathbb{R}^m$, or, taking the contrapositive, if $T(u) = T(v)$ in $\mathbb{R}^m$, then $u = v$ in $\mathbb{R}^n$.

We have already noted that for every linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, there exists an $m \times n$ matrix $A$ such that $T(x) = Ax$. Let’s see what this says about the matrix $A$ and how this relates to previously learned concepts (hint: Section 1.7).

Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation.

1. Let $u$ and $v$ be vectors in $\mathbb{R}^n$ such that $T(u) = T(v)$. Show that $T(u - v) = 0$.

2. Based on your work in the previous question, we see that whenever $T(u) = T(v)$, then there exists a vector $w$, namely $w = u - v$, such that $T(w) = 0$. Thus, if we want to decide if $T$ is one-to-one, it makes sense to look at the set of solutions to the equation $T(x) = 0$.

   (a) We have previously seen that the equation $T(x) = 0$ has at least one solution. What is it?

   (b) If $T(w) = 0$ for some nonzero vector $w$ in $\mathbb{R}^n$, is $T$ one-to-one? Why or why not?

   (c) If $T$ is one-to-one, how many solutions are there to the equation $T(x) = 0$?

   (d) If $A$ is the standard matrix for $T$ and $T$ is one-to-one, how many solutions are there to the equation $Ax = 0$?
Since solving the equation $T(x) = 0$ is integral to determining whether or not $T$ is one-to-one, we make the following definition:

**Definition:** The null space of $T$ is the set of solutions to the equation $T(x) = 0$, that is, null space $T = \{ x \in \mathbb{R}^n \mid T(x) = 0 \}$.

**Note:** finding the null space of a linear transformation $T$ with standard matrix $A$ is the same as finding the general solution to the equation $Ax = 0$.

3. Using your work in the previous question, we have discovered that if $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a one-to-one linear transformation with $m \times n$ standard matrix $A$, then the only solution to $Ax = 0$ is $x = 0$. Since you have seen this statement before, complete the following:

A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with standard matrix $A$ is one-to-one if and only if

(a) null space $T = \ldots$
(b) The only solution to $Ax = 0$ is $\ldots$.
(c) rank $A = \ldots$
(d) nullity $A = \ldots$
(e) Columns of $A$ are $\ldots$ $\ldots$.
(f) There is a pivot position in every $\ldots$ of $A$.
(g) $Ax = b$ has at most $\ldots$ solution for every $b \in \mathbb{R}^m$.
(h) The columns of $\text{rref}(A)$ are distinct $\ldots$ $\ldots$.

4. Consider the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(x) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix}$.

(a) Find the standard matrix $A$ for $T$.

(b) Find the null space of $T$. Is $T$ one-to-one? Why or why not?
5. Consider the linear transformation \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) defined by \( T(x) = T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 - 4x_2 \\ 2x_1 + x_3 \end{bmatrix} \).

(a) Find the standard matrix \( A \) for \( T \).

(b) Find the null space of \( T \). Is \( T \) one-to-one? Why or why not?

6. Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation. Mark each of the following True or False and explain.

(a) If \( n > m \), then \( T \) is not one-to-one.

(b) If \( n \leq m \), then \( T \) is one-to-one.

(c) If \( T \) is one-to-one and onto, then \( m = n \).