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Exercise for April 16, 2005

You are given that X and Y both have the same uniform distribution on [0, 1], and are independent. $U = X + Y$ and $V = \frac{X}{X+Y}$. Find the joint probability density function of (U, V) evaluated at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

A. 0 B. $\frac{1}{4}$ C. $\frac{1}{3}$ D. $\frac{1}{2}$ E. 1

Solution.
We have $X = UV$, and $Y = U - UV$. Therefore,

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix} = -uv - u + uv = u.$$

This implies that

$$f_{U,V}(u,v) = f_{X,Y}\left(x(u,v), y(u,v)\right) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1 \cdot 1 \cdot |u| = u.$$

This density at $\left(\frac{1}{2}, \frac{1}{2}\right)$ equals $\frac{1}{2}$.

Answer D.