Tate cohomology often fails to detect null homotopy

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The Problem

G is a p-group and k is a field of characteristic p.

$f, g\colon M \to N$ between finite dimensional kG-modules are **homotopic** if their difference $(f - g)$ factors through a projective.

$f\colon M \to N$ is **null-homotopic** if it factors through a projective.

Tate cohomology functor:

$$\begin{aligned} \widehat{H}^*(G, -): \quad \mathrm{mod}(kG) &\longrightarrow \text{Graded vector spaces over } k. \\ M &\longmapsto \widehat{H}^*(G, M) \end{aligned}$$

Problem: For which p-groups G, does the Tate cohomology functor $\widehat{H}^*(G, -)$ detect null-homotopic maps between finite-dimensional modules?

The stable module category – $\mathrm{stmod}(kG)$

- **Objects:** finite-dimensional kG-modules.
- **Morphisms:** homotopy classes of kG-linear maps.

Note:

1. A module M is projective if and only if it is the zero object in $\mathrm{stmod}(kG)$.
2. A map of kG-modules is null homotopic if and only if it is the zero map in $\mathrm{stmod}(kG)$.

$\mathrm{stmod}(kG)$ is a **triangulated category**:

1. $\Omega(M) := \ker(P_M \twoheadrightarrow M)$
2. exact triangles are given by the short exact sequences of kG-modules.

Fact: $\widehat{H}^i(G, M) \cong \underline{\mathrm{Hom}}_{kG}(\Omega^i k, M)$ for all i.

Definition: A *ghost* in stmod(kG) is a map $\phi : X \longrightarrow Y$ that is invisible to Tate cohomology, i.e., for all i

$$\underline{\text{Hom}}(\Omega^i k, X) \longrightarrow \underline{\text{Hom}}_{kG}(\Omega^i k, Y) \quad \text{is the zero map.}$$

Theorem [BCCM06]

The Tate cohomology functor detects all null-homotopic maps between finite-dimensional kG-modules if and only if G is either C_2 or C_3.

Equivalently, there are no non-trivial ghosts in stmod(kG) if and only if G is isomorphic to C_2 or C_3.

Proof Sketch:

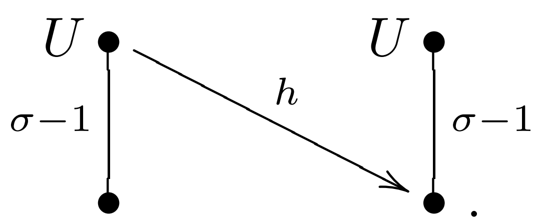
Step 1: If H is a subgroup of G, and if $\widehat{H}^*(H, -)$ fails to be faithfull on stmod(kH), then so will be the functor $\widehat{H}^*(G, -)$ on stmod(kG).

$$\phi\colon M \to N \text{ is a non-trivial ghost in stmod}(kH),$$
$$\Downarrow$$
$$\phi{\uparrow}^G\colon M{\uparrow}^G \to N{\uparrow}^G \text{ is a non-trivial ghost in stmod}(kG).$$

Step 2: The functor $\widehat{H}^*(C_{p^r}, -)$ fails to detect some null-homotopic map if $p^r \geq 4$.

$C_{p^r} = \langle \sigma \rangle$ and M be a cyclic module of length two generated by U. (so we have $(\sigma - 1)^2 U = 0$.)

$h\colon M \to M$ multiplies by $\sigma - 1$:



It is not hard to see that h is a non-trivial (uses $p^r \geq 4$) ghost.

Step 3: The functor $\widehat{H}^*(C_p \oplus C_p, -)$ also fails to detect some null-homotopic map.

$G = C_p \oplus C_p = \langle \alpha \rangle \oplus \langle \beta \rangle$

k_α be the trivial $k\langle \alpha \rangle$-module

The map $k_\alpha{\uparrow}^G \xrightarrow{\beta - 1} k_\alpha{\uparrow}^G$ is a non-trivial ghost.

Non-triviality can be shown by restricting to H.

Lemma: Let G be a finite p-group and let x be a central element in G. Then for any kG-module M, the map $x - 1\colon M \to M$ is a ghost.

Consider the commutative diagram

$$\begin{array}{ccc} \Omega^n k & \xrightarrow{f} & M \\ {\scriptstyle x-1}\big\downarrow & & \big\downarrow{\scriptstyle x-1} \\ \Omega^n k & \xrightarrow[f]{} & M. \end{array}$$

Note that $x - 1$ acts trivially on k, so functoriality of Ω shows that the left vertical map is stably trivial.

Step 4: (Easy exercise) If G is not C_2 or C_3, then G either has a cyclic subgroup of order at least four or a subgroup of the form $C_p \oplus C_p$.

Step 5: (If part) Assume $G = C_2$ or C_3.

$$kC_2 \cong k[x]/x^2 \quad \text{and} \quad kC_3 \cong k[x]/x^3.$$

Every finite-dimensional *projective-free* kG-module is a direct sum of **syzygies** $\Omega^i k$ of k,

It follows that every null-homotopic map is detected by the functors $\underline{\mathrm{Hom}}_{kG}(\Omega^* k, -)$.

Ghost number

The **ghost number** of kG is the smallest integer l such that the composition of any l ghosts between finite-dimensional kG-modules is null-homotopic, i.e., trivial in $\mathrm{stmod}(kG)$

By our theorem the ghost number of G is 1 if and only if G is C_2 or C_3.

Example: Consider $V_4 = C_2 \oplus C_2$. We know that there is a ghost that is not null-homotopic. However, the composition of any two ghosts is null-homotopic.

$$0 \longrightarrow M^G(= \oplus k) \longrightarrow M \longrightarrow M_G(= \oplus k) \longrightarrow 0$$

Moral: In order to compute the ghost number of kG, it is important to know how the various finite-dimensional indecomposable representations of G are obtained by gluing (using short exact sequences) the syzygies $(\Omega^i k)$ of k in the *most economical way.*

Here are some computations and bounds of ghost numbers.

Theorem [CCM06]

1. The ghost number for kC_{p^r} is $\lceil (p^r - 1)/2 \rceil$.
2. The ghost number for kG is at most the nilpotency index of $J(kG) - 1$.
3. If G is an abelian p-group with C_2 as summand, then the ghost number of kG is equal to the nilpotency index of $J(kG) - 1$.
4. The ghost number of $k(C_2)^l$ is l.
5. If G is a 2-group that sits in an extension:

   $$1 \longrightarrow C_{2^{n-1}} \longrightarrow G \longrightarrow C_2 \longrightarrow 1$$

   then the ghost number of G is at most 2^{n-1}

Thank You