A new perspective on groups with periodic cohomology

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Outline

- Groups with periodic cohomology
- Stable module category and Tate cohomology
- The finite generation problem
- Some related results

Groups with periodic cohomology

Let G be a finite group and let p be a prime.

The group cohomology $\mathrm{H}^*(G, \mathbb{F}_p)$ of G is just the ordinary mod–p cohomology of the classifying space BG of G.

We say G has periodic cohomology if for some integer $d > 0$

$$\mathrm{H}^{*+d}(G, \mathbb{F}_p) \cong \mathrm{H}^*(G, \mathbb{F}_p)$$

Artin–Tate(1952): G has periodic cohomology if and only if the Sylow p–group of G is either a cyclic group or a generalised Quaternion group.

Swan(1960): G has periodic cohomology for all p if and only if G acts freely on a finite CW complex which has the homotopy type of a sphere.

The stable module category

G is a finite group and k is a field of characteristic $p > 0$ which divides the order of G.

$f, g \colon M \to N$ are homotopic if their difference $f - g$ factors through a projective.

The stable module category – $\mathrm{stmod}(kG)$

- Objects: finitely generated kG–modules.
- Morphisms: homotopy classes of kG–linear maps.

$\mathrm{stmod}(kG)$ is a triangulated category:

1. $\Omega(M) := \ker(P_M \twoheadrightarrow M)$
2. exact triangles are given by the short exact sequences of kG–modules.

Tate cohomology

Tate cohomology functor:

$$\begin{array}{rccl} \widehat{\mathrm{H}}^*(G,-): & \mathrm{stmod}(kG) & \longrightarrow & \text{graded vector spaces over } k. \\ & M & \longmapsto & \widehat{\mathrm{H}}^*(G,M) \cong \underline{\mathrm{Hom}}_{kG}(\Omega^* k, M). \end{array}$$

Tate cohomology ring of G is $\widehat{\mathrm{H}}^*(G,k)$. This is a graded commutative ring. Cup products are given by composition of maps!

$\widehat{\mathrm{H}}^*(G,M)$ is naturally a module over $\widehat{\mathrm{H}}^*(G,k)$.

Tate cohomology in positive degrees is just the ordinary cohomology.

Tate duality: Ties up positive and negative cohomologies

$$\langle\ ,\ \rangle : \widehat{\mathrm{H}}^i(G,k) \otimes \widehat{\mathrm{H}}^{-i-1}(G,k) \xrightarrow{\cup} \widehat{\mathrm{H}}^{-1}(G,k) \cong k$$

The finite generation problem

Let M be a finitely–generated kG–module.

Evens–Venkov(1962): $\mathrm{H}^*(G, M)$ is finitely generated as a graded module over $\mathrm{H}^*(G, k)$.

Question: Is it true that $\widehat{\mathrm{H}}^*(G, M)$ is finitely generated as a graded module over $\widehat{\mathrm{H}}^*(G, k)$?

Answer: *resounding NO!*

Weaker question: Which groups G have the property that $\widehat{\mathrm{H}}^*(G, M)$ is finitely generated for all M?

Partial Answer: Groups with periodic cohomology have this property. (Proof is Clear.)

Answer to the weaker question

Theorem (Carlson–C–Minac, 2007)
$\widehat{\mathrm{H}}^*(G, M)$ *is finitely generated for all finitely generated* kG*–modules* M *if and only if* G *has periodic cohomology.*

Combining this with Swan's result, we get:

Corollary (Carlson–C–Minac, 2007)
$\widehat{\mathrm{H}}^*(G, M)$ *is finitely generated for all* M *and all* p *if and only if* G *acts freely on a finite CW complex which has the homotopy type of a sphere.*

Proof Strategy

It suffices to show that if G has non–periodic cohomology then there exists a kG–module whose Tate cohomology is not finitely generated.

We proved a much stronger statement:

Theorem (Carlson–C–Minac, 2007)

Suppose that G has non–periodic cohomology. Let M be any non–projective periodic kG–module. Then $\widehat{\mathrm{H}}^*(G, \mathrm{End}_k M)$ is not finitely generated as a $\widehat{\mathrm{H}}^*(G, k)$-module.

M is periodic if $\Omega^t M \cong M$ in $\mathrm{stmod}(kG)$ for some t.

Proof uses Chouinard's theorem, cyclic shifted subgroups, a result on negative cohomology by Benson and Carlson.

Sketch proof

Chouinard(1976): There exists a maximal elementary abelian p–subgroup $E = \langle x_1, \ldots, x_n \rangle$ such that the restriction M_E is not projective.

There exists an element $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$ and a corresponding cyclic shifted subgroup $\langle u_\alpha \rangle$,

$$u_\alpha \ := \ 1 \ + \ \sum_{i=1}^{n} \alpha_i (x_i - 1)$$

such that the restriction of M to $\langle u_\alpha \rangle$ is not projective.

Cohomological interpretation: The identity map $\mathrm{Id}_M : M \longrightarrow M$ does not factor through a projective $k\langle u_\alpha \rangle$–module. $\widehat{\mathrm{H}}^0(\langle u_\alpha \rangle, \mathrm{End}_k M) \cong \underline{\mathrm{Hom}}_{k\langle u_\alpha \rangle}(k, \mathrm{End}_k M)$, so we have a non–zero class in $\widehat{\mathrm{H}}^0$ which is represented by $1 \in k \mapsto \mathrm{Id}_M$.

Two crucial ingredients

Let t be the period of M so that $\Omega^{mt}(M) \cong M$ for all m.

I. By periodicity there exists elements

$$\zeta_m \in \widehat{\mathrm{H}}^{mt}(G, \mathrm{End}_k M)$$

such that ζ_m is not zero on restriction to $\langle u_\alpha \rangle$.

II. Benson–Carlson(1992): Since G has p–rank at least 2, the restriction map

$$\mathrm{res}_{G,\langle u_\alpha \rangle} : \widehat{\mathrm{H}}^d(G, k) \longrightarrow \widehat{\mathrm{H}}^d(\langle u_\alpha \rangle, k)$$

is the zero map if $d < 0$.

Suppose that $\widehat{\mathrm{H}}^*(G, \mathrm{End}_k M)$ is finitely generated.

Then there exist generators $\mu_1, \ldots, \mu_r$ in degrees $d_1, \ldots, d_r$, respectively, in $\widehat{\mathrm{H}}^*(G, \mathrm{End}_k M)$.

Now choose an integer m such that $mt < \min\{d_i\}$.

Then we must have that

$$\zeta_m = \sum_{i=1}^{r} \gamma_i \mu_i$$

for some $\gamma_i \in \widehat{\mathrm{H}}^{mt-d_i}(G, k)$.

I. $\implies \mathrm{res}_{G,\langle u_\alpha \rangle}(\zeta_m) \neq 0$.

II. $\implies \mathrm{res}_{G,\langle u_\alpha \rangle}(\gamma_i) = 0 \ \forall \ i$ (since $mt - d_i < 0$).

QED.

Our Conjecture

Definition: The support variety $V_G(M)$ of M is defined as follows.

$$V_G(M) := \mathrm{Var}\left[\mathrm{Ker}\colon\ \mathrm{Ext}^*_{kG}(k,k) \xrightarrow{\otimes M} \mathrm{Ext}^*_{kG}(M,M)\right]$$

Support varieties are closed homogeneous subvarieties of the maximal ideal spectrum for the group cohomology ring.

Conjecture: For an indecomposable non–projective kG–module M, $\widehat{\mathrm{H}}^*(G,M)$ is finitely generated over $\widehat{\mathrm{H}}^*(G,k)$ if and only if

$$V_G(M) = V_G(k).$$

Affirmative result

The following result supports our conjecture.

Theorem (Carlson–C.–Mináč, 2007)

Let G be a group with the property that product of any two elements in negative cohomology is trivial. If $V_G(M) \subseteq V_G(\zeta)$ for some regular element ζ in the cohomology ring, then $\widehat{\mathrm{H}}^*(G, M)$ is not finitely generated over $\widehat{\mathrm{H}}^*(G, k)$.

Example: For G as above, let ζ be a regular element in $\mathrm{H}^*(G, k)$.

$$L_\zeta := \mathrm{Ker}(\zeta : \Omega^{|\zeta|} \longrightarrow k)$$

$V_G(L_\zeta) = V_G(\zeta)$ and consequently $\widehat{\mathrm{H}}^*(G, L_\zeta)$ is not finitely generated.

Modules with finitely generated Tate cohomology

Theorem (Carlson–C.–Mináč, 2007)

Let G be a group with non–periodic cohomology. Let ζ be an element in the Tate cohomology ring of G such that

$$\mathrm{Dim}\ \ \mathrm{Im}[\zeta\colon \widehat{\mathrm{H}}^*(G,k) \longrightarrow \widehat{\mathrm{H}}^*(G,k)] < \infty$$

Then the Tate cohomology of the module M represented by ζ is finitely generated over $H^*(G,k)$.

If ζ belongs to $\widehat{\mathrm{H}}^{n+1}(G,k)$, then there exists a module M which sits in:

$$\zeta\colon 0 \longrightarrow k \longrightarrow M \longrightarrow \Omega^n k \longrightarrow 0$$

The middle terms of Auslander–Reiten sequences ending in k are a good source of examples.

Thank You