Stable Homotopy Theory – A gateway to modern mathematics.

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Plan of the talk

1. Introduction to stable homotopy theory
2. Some major and global problems in the field
3. Axiomatic stable homotopy theory
4. Some results from my thesis
5. Recent work with Minac and Christensen

At one time it seemed as if homotopy theory was utterly without system; now it is almost proved that systematic effects predominate.

— Frank Adams (1988).

1 Introduction

Some early evidence for system.

- **Freudenthal suspension theorem:** If X and Y are any two finite CW-complexes, then the (reduced) suspension map

$$[X, Y] \longrightarrow [\Sigma X, \Sigma Y]$$

  is an isomorphism if $\dim(X) \leq 2\,\mathrm{Conn}(Y)$.

- **Natural group structures:** $[X, Y]$ has a natural group structure if $\dim(X) \leq 2\,\mathrm{Conn}(Y)$.

- **Spaces with a homotopy type of loop space:** Suppose $\pi_i(X) = 0$ for $i > 2\,\mathrm{Conn}(X)$. Then X has the homotopy type of a loop space.

SUMMARY

In a range of dimensions and connectivities of the spaces, homotopy theory has many interesting properties; these properties do not hold outside this range.

This leads to ***Stable Homotopy Theory*** – homotopy theory in the "stable range".

Stable homotopy theory is sometimes also known as the *"Homotopy theory of negative dimensional spheres" !*

$$\text{Spaces} : \text{Spectra} \quad \longleftrightarrow \quad \mathbb{N} : \mathbb{Z}$$

Spanier-Whitehead category

We need a home for stable homotopy theory! – a category which can isolate stable phenomena in homotopy theory.

Objects: Ordered pairs (X, n) where X is a CW-complex and n is any integer.

Morphisms: $\{(X, n), (Y, m)\} := \operatorname{colim}_k [\Sigma^{n+k} X, \Sigma^{m+k} Y]$.

In particular, when $X = S^0$ and $m = 0$, this gives the n^{th} stable homotopy group of Y, denoted $\pi_n^s(Y)$.

Suspension: $\Sigma(X, n) := (X, n+1) (\cong (\Sigma X, n))$ and $\Sigma^{-1}(X, n) := (X, n-1)$.

So we have inverted the suspension functor in this new category!

Good News: This is a very good home for finite stable homotopy theory! In fact, this is the ideal stabilisation of finite dimensional CW-complexes

Bad News: Does not work for infinite dimensional complexes. This category is too small; does not have arbitrary coproducts.

Good News: This category can be repaired so that it has all the desired properties. The resulting category is called the "Stable homotopy category of spectra"

Bad News: The construction of the stable homotopy category is **incredibly complicated** !!

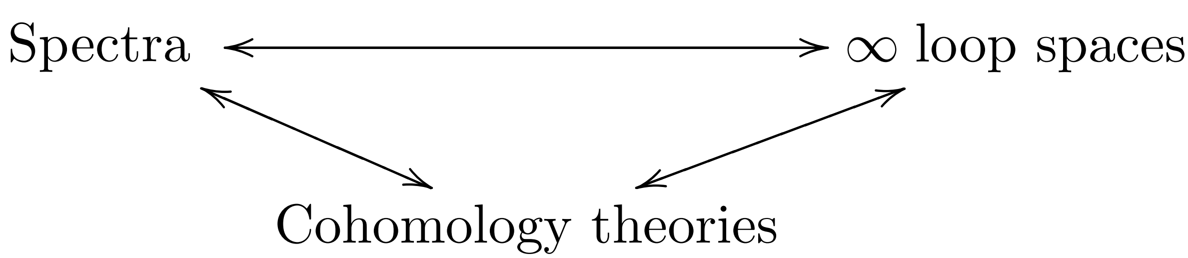
Good News: One does not have to worry about the technicalities involved in the construction for the most part.

Examples

Spectra represent generalised cohomology theories on CW complexes.

- Singular cohomology
- Complex K-theory
- Complex cobordism

The study of the following 3 subjects in essentially equivalent.



2 Global problems

When is a map $f : X \to Y$ between CW-complexes stably null-homotopic? ($f \simeq_s 0 \Leftrightarrow \Sigma^t f \simeq 0$ for t large.)

The Generating Hypothesis: (Peter Freyd - 1966)

A map $f : X \to Y$ between finite CW-complexes is stably hull-homotopic if $\pi_*^s(f) = 0$.

This conjecture is false for infinite dimensional complexes: A non-zero positive degree element in the mod-2 Steenrod algebra represents a *non-trivial* map $\phi : \Sigma^d H\mathbb{F}_2 \to H\mathbb{F}_2$. But $\pi_*^s(\phi)$ is clearly zero.

Only some partial (affirmative) results are known when the target is a sphere (Devinatz - 1990).

Nilpotence detection: When is a self map $f : \Sigma^t X \to X$ of a CW-complex stably nilpotent?

f is *stably nilpotent* if some iterate

$$\Sigma^{kt} X \to \Sigma^{(k-1)t} X \to \cdots \Sigma^t X \to X$$

of f is stably null-homotopic.

Theorem 1. *(Nishida -1973) A self-map $\Sigma^t S^n \to S^n$, for $t > 0$, is stably nilpotent!*

The Nipotence Theorem

Is there a generalised homology theory $E_*(-)$ which can detect stable nilpotence of self maps?

Theorem 2. *(Devinatz-Hopkins-Smith: 1988) There is a generalised homology theory $MU_*(-)$ (*Complex Bordism*) which detects stable nilpotence. More precisely, if X any finite CW-complex, a self map $f : \Sigma^d X \to X$ is stably nilpotent if and only if the endomorphism $MU_*(f) : \Sigma^k MU_*(X) \to MU_*(X)$ is nilpotent.*

This is a remarkable theorem which has laid the foundation for much of modern homotopy theory. $MU_*(-)$ is often computable, so this theorem is quite powerful.

Existence of periodic maps:

A map between CW-complexes is stably periodic if it is not stably nilpotent. Does every CW-complex admit a stable periodic self-map?

Why do we care about such maps? Such maps help us to detect new families in the stable homotopy groups of spheres!

Adams showed that for large n there is a stable periodic map $\Sigma^q M(p) \xrightarrow{A} M(p)$ $(q = 2(p-1))$ of the Moore space $(M(p) := S^n \bigcup_p e^{n+1})$ which induces isomorphism in K-theory. This gives a "systematic family" in the stable homotopy groups of spheres - first constructed by Toda.

$$
\begin{array}{ccccccccc}
\Sigma^{iq}M(p) & \xrightarrow{\Sigma^{iq}A} & \cdots & \longrightarrow & \Sigma^{2q}M(p) & \xrightarrow{\Sigma^{2q}A} & \Sigma^{q}M(p) & \xrightarrow{A} & M(p) \\
\uparrow & & & & & & & & \downarrow \\
S^{n+iq} & & & & \overset{\alpha_i}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & & & \dashrightarrow & S^{n+1}
\end{array}
$$

Note that α_i belongs to $\pi^s_{iq-1}(S^0)$ for all i.

Is the same true for any arbitrary finite CW-complex?

Morava K-theories: Fix a prime p. For $n \geq 0$, there are Morava K-theories which define generalised homology theories $K(n)_*(-)$. For example $K(0)_*(X) \cong H(X;\mathbb{Q})$, $K(\infty)_*(X) := H_*(X;\mathbb{F}_p)$.

A finite CW-complex is of type-n (at p) if $K(n)_i(X) = 0$ for $i < n$ and $K(n)_*(X) \neq 0$.

The periodicity theorem

Theorem 3. *(Hopkins-Smith: 1998) Let X be a finite CW-complex of type $n > 0$ at prime p. Then there is a stable periodic self-map*

$$f : \Sigma^{d+i} X \to X \textit{ for some } i >> 0$$

such that $K(t)_* f$ is an isomorphism if $t = n$, and 0 for $t > n$.

So we have lots of periodic maps! — one of every type-n complex (Mitchell showed their existence in 1985). These will help us detect new families in $\pi_*^s(S)$.

3 Axiomatic stable homotopy theory

There are a bunch of axioms which define stable homotopy theory.

These axioms are similar to the Eilenberg-Steenrod axioms which define singular homology theory.

One of the axioms which define stable homotopy theory is:

Finiteness axiom: The full subcategory of finite objects is equivalent to the Spanier-Whitehead category consisting of finite CW-complexes.

One gets *generalised stable homotopy theories* by dropping this finiteness axiom – this is analogous to dropping the dimension axiom ($E_i(*) = 0$ for $i \neq 0$) from the Eilenberg-Steenrod axioms to obtain generalised homology theories.

This allows us to “do stable homotopy theory” in algebra, representation theory and many more..

Modern View point(Hovey-Palmieri-Strickland 1997)

A *stable homotopy category* is a “sufficiently well-behaved” triangulated category - These are categories that are formally similar to the stable homotopy category of spectra. E.g. The derived category of a ring, stable module category of a group algebra etc.

Triangulated categories

Triangulated categories are additive categories with:

1. Suspension $\Sigma : \mathcal{T} \xrightarrow{\simeq} \mathcal{T}$,
2. Exact triangles $A \to B \to C \to \Sigma A$,
3. Axioms.

Stable homotopy categories have more structure:

1. Compatible smash product $\wedge : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$,
2. Sphere object S ($S \wedge X \cong X$),
3. Brown representability + more.

Examples.

1. $K(R)$ – Homotopy category of R.
2. $D(R)$ – Derived category of R.
3. $\mathrm{StMod}(kG)$ – Stable module category of kG.
4. $K(\mathrm{Proj}\ B)$ – Chain homotopy category of projective B-modules.
5. $\mathcal{S}$ – Stable homotopy category of spectra.

An algebraic Nilpotence theorem

Theorem 4. *(Hopkins - 1985) Let* $f : X_\bullet \to Y_\bullet$ *be a self-map of a perfect complexes. Then* f *is tensor-nilpotent (i.e.,* $f^{\otimes n} = 0$ *for some integer* n*) if and only if*

$$f \otimes K(p) : X_\bullet \otimes K(p) \to Y_\bullet \otimes K(p)$$

is zero for all primes p*.*

$K(p)$ is the fraction field of the domain R/p - they play the role of the Morava K-theories.

There is a similar nilpotence theorem due to Jon Carlson for the stable module category.

Theorem 5 (Hopkins)**.** *Let X and Y be finite p-local spectra. Then Y can be generated from X using cofibrations and retractions if and only if*

$$\mathrm{Supp}(Y) \subseteq \mathrm{Supp}(X).$$

$$\mathrm{Supp}(A) = \{n : K(n)_* A \neq 0\} \leftarrow \textit{chromatic support.}$$

Theorem 6 (Hopkins)**.** *Let X and Y be perfect complexes. Then Y can be generated from X using cofibrations and retractions if and only if*

$$\mathrm{Supp}(Y) \subseteq \mathrm{Supp}(X).$$

$$\mathrm{Supp}(A) = \{p \in \mathrm{Spec}(R) : A \otimes R_{(p)} \neq 0\} \leftarrow \textit{homology support.}$$

Refinements of Chromatic Towers for spectra

Theorem 7 (Hopkins-Smith)**.** *The thick subcategories of $\mathcal{F}_p$ are precisely the acyclics of the Morava K-theories.*

$$\mathcal{C}_n = \{X \in \mathcal{F}_p :\ K(n-1)_* X = 0\},$$

$$\cdots \subsetneq \mathcal{C}_{n+1} \subsetneq \mathcal{C}_n \subsetneq \mathcal{C}_{n-1} \subsetneq \cdots \subsetneq \mathcal{C}_2 \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0 \ (= \mathcal{F}_p).$$

Define Euler characteristic functions:

$$\mathcal{C}_0 \qquad \chi_0(X) = \sum_i (-1)^i \dim_{\mathbb{Q}} H\mathbb{Q}_i(X),$$

$$\mathcal{C}_n \qquad \chi_n(X) = \sum_i (-1)^i \log_p |BP\langle n-1\rangle_i X|.$$

Theorem 8. *There is a family $\{\mathcal{C}_n^k\}_{k\geq 0}$ of triangulated subcategories of $\mathcal{F}_p$ such that*

$$\cdots \subsetneq \mathcal{C}_{n+1} \subsetneq \mathcal{C}_n^k \subsetneq \mathcal{C}_n \subsetneq \cdots \subsetneq \mathcal{C}_2 \subsetneq \mathcal{C}_1^k \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0^k \subsetneq \mathcal{C}_0$$

$$\textit{where}\ \ \mathcal{C}_n^k = \{X \in \mathcal{C}_n : \chi_n(X) \equiv 0 \mod l_n k\}.$$

Moreover when $n = 0$ or 1, every dense triangulated subcategory of $\mathcal{C}_n$ is $\mathcal{C}_n^k$ for some k.

Corollary 1. *Let X be a type-0 spectrum and Y an arbitrary finite p-local spectrum. Then Y can be generated by X via cofibrations if and only if $\chi_0(X)$ divides $\chi_0(Y)$.*

Corollary 2. *Let X be a type-1 spectrum and Y an arbitrary finite p-torsion spectrum. Then Y can be generated by X via cofibrations if and only if $\chi_1(X)$ divides $\chi_1(Y)$.*

Note: $\chi_1(M(p)) = 1$ and $\chi_1(M(p^2)) = 2$. So $M(p)$ cannot be generated by iterated cofiberings of $M(p^2)$.

Corollary 3. *Every dense triangulated subcategory of $\mathcal{F}_p$ is a triangulated ideal.*

$D^b(\text{proj } R)$ – Perfect complexes

Artin rings

Theorem 9. *Let X and Y be perfect complexes over an Artin ring. Then Y can be generated from X using* <u>*cofibrations*</u> *if and only if*

- $\text{Supp}(Y) \subseteq \text{Supp}(X)$, *and*
- $\Lambda_p(X)$ *divides* $\Lambda_p(Y)$ *for all* $p \in \text{Supp}(X)$,

$$\Lambda_p(X) = \sum_i (-1)^i \dim_{R/p} H_i(X \otimes R/p).$$

Krull-Schmidt decompositions

A collection $\{D_i\}_{i \in I}$ of thick subcategories is a *Krull-Schmidt* decomposition of a thick subcategory $\mathcal{D}$ if

1. $\mathcal{D} = \coprod_{i \in I} \mathcal{D}_i$
2. $\mathcal{D}_i \cap \mathcal{D}_j = 0$ for $i \neq j$
3. $\mathcal{D}_i$ are indecomposable
4. Uniqueness

Example. $\mathcal{T}$ = finite torsion spectra.

$\mathcal{T}_p$ = finite p-torsion spectra.

$$\mathcal{T} = \coprod_p \mathcal{T}_p.$$

Theorem 10. *The thick subcategories of compact objects admit Krull-Schmidt decompositions in the following categories*

- *Stable homotopy category*
- *Derived categories of noetherian rings*
- *Stable module categories of finite dimensional co-commutative Hopf algebras.*

Decompositions in $D^b(\text{proj } R)$

Let R be any commutative ring such that:

1. Every open subset of $\text{Spec}(R)$ is compact.
2. $\text{Spec}(R)$ satisfies the d.c.c.

Examples. Noetherian rings, and rings with finitely many primes.

Theorem 11. *Every thick subcategory of* $D^b(\text{proj } R)$ *admits a Krull-Schmidt decomposition.*

Proof sketch: We use the Hopkins-Neeman theorem which gives a bijection between thick subcategories $\mathcal{A}$ of $D^b(\text{proj } R)$ and specialisation closed subsets S of $\text{Spec}(R)$

$$S = \bigcup_{X \in \mathcal{A}} \text{Supp}(X).$$

$$
\begin{array}{ccc}
\mathcal{A} & \longleftrightarrow & S \\
\Vert & & \Vert \\
\coprod \mathcal{A}_i & \longleftrightarrow & \bigcup S_i
\end{array}
$$

Want a decomposition $S = \bigcup S_i$ of S into indecomposable specialisation closed subsets.

Define a graph G_S: Vertices are the minimal primes of S.

Adjacency: $p \sim q \Leftrightarrow V(p) \cap V(q) \neq \emptyset$.

C_i – connected components of G_S.

$$S_i := \bigcup_{p \in C_i} V(p).$$

$$\mathcal{A}_i := \{X \in D^b(\text{proj } R) : \text{Supp}(X) \subseteq S_i\}.$$

$$\mathcal{A} \cong \coprod_{i \in I} \mathcal{A}_i.$$

Some corollaries

Corollary 4. *Every perfect complex X over R admits a unique splitting into perfect complexes,*

$$X \cong \bigoplus_{i \in I} X_i$$

such that the supports of the X_i are pairwise disjoint and indecomposable.

Corollary 5. *A noetherian ring R is local if and only if every thick subcategory of $D^b(\mathrm{proj}\ R)$ is indecomposable.*

Corollary 6. *Let $\mathcal{A} = \coprod_{i \in I} \mathcal{A}_i$ be a Krull-Schmidt decomposition of a thick subcategory of perfect complexes. Then,*

$$K_0(\mathcal{A}) \cong \bigoplus_{i \in I} K_0(\mathcal{A}_i).$$

4 Ghosts in modular representation theory

A ghost is a map between finite dimensional representations of a group G that is invisible in Tate cohomology.

Generating hypothesis (GH) for a finite group G is the statement that there are no non-trivial ghosts between finite dimensional representations of G.

Theorem 12. *Let G be a finite group whose Sylow p-subgroups are not cyclic. If the trivial representation k is periodic, then GH fails in* $\operatorname{stmod}(kG)$*. In particular, the GH fails in* $\operatorname{stmod}(kG)$ *whenever the Sylow 2-subgroups of G are generalised Quaternion group.*