A BRAVE NEW WORLD IN HOMOTOPY THEORY

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At one time it seemed as if homotopy theory was utterly without system; now it is almost proved that systematic effects predominate. — Frank Adams (1988).

1 Motivation

Some early evidence for system.

1. **Stability of reduced homology:** For any CW-complex X, $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(\Sigma X)$ for all n.

2. **Freudenthal suspension theorem:** If X and Y are any two CW-complexes, then the (reduced) suspension map

   $$[X, Y] \longrightarrow [\Sigma X, \Sigma Y]$$

   is an isomorphism if $\dim(X) \leq 2\,\mathrm{Conn}(Y)$.

3. **Natural group structures:** $[X, Y]$ has a natural group structure if $\dim(X) \leq 2\,\mathrm{Conn}(Y)$.

4. **Spaces with a homotopy type of loop space:** Suppose $\pi_i(X) = 0$ for $i > 2\,\mathrm{Conn}(Y)$. Then X has the homotopy type of a loop space.

5. **Merging of fibrations and cofibrations:** In a certain range, the fibre and cofibre of a map have, up to a shift, the same homotopy groups.

SUMMARY: In a range of dimension and connectivities of the spaces, homotopy theory has many interesting properties; these properties do not hold outside this range.

This leads to *Stable homotopy theory* – homotopy theory in the "stable range"

2 Spanier-Whitehead category

We need a home for stable homotopy theory! i.e., a category which can isolate stable phenomena in homotopy theory.

Objects: Ordered pairs (X, n) where X is a CW-complex and n is any integer.

Morphisms: $\{(X, n), (Y, m)\} := \operatorname*{colim}_k\, [\Sigma^{n+k}X, \Sigma^{m+k}Y]$.

In particular, when $X = S^0$ and $n = 0$, this gives the m^{th} stable homotopy group of Y, denoted $\pi_m^s(Y)$.

Suspension: $\Sigma(X, n) := (X, n+1)$ and $\Sigma^{-1}(X, n) := (X, n-1)$.

So we have inverted the suspension functor in this new category!

Good News: This is a very good home for finite stable homotopy theory! In fact, this is the ideal stabilisation of finite dimensional CW-complexes

Bad News: Does not work for infinite dimensional complexes. This category is too small; does not have arbitrary coproducts.

Good News: This category can be repaired so that it has all the desired properties. The resulting category is called the “Stable homotopy category of spectra”

Construction is quite involved. Enlarge the category of spaces to get “spectra”.

$$\begin{aligned} \text{Spaces} &\hookrightarrow \text{Spectra} \\ X &\longmapsto (X, 0) \end{aligned}$$

First successful model was given by Boardman. Later Adams and others have obtained other models.

Homotopy: Maps $f : X \to Y$ and $g : X \to Y$ are *homotopic* ($f \simeq g$) if there is a map $H : X \times [0,1] \to Y$ such that $H|_{X\times 0} = f$ and $H|_{X\times 1} = g$.

Stable Homotopy: Maps $f : X \to Y$ and $g : X \to Y$ are *Stably homotopic* ($f \simeq_s g$) if $\Sigma^k f$ is homotopic to $\Sigma^k g$ for some k.

Example: The quotient map from Torus to the Sphere ($\pi : S^1 \times S^1 \to S^1 \wedge S^1$) obtained by collapsing the wedge $S^1 \vee S^1$ to a point is *not homotopic* to the constant map. However, it is *stably homotopic* to the constant map.(why?)

3 Global problems in stable homotopy theory

When is a map $f : X \to Y$ between CW-complexes stably null-homotopic? ($f \simeq_s 0$) This is an extremely hard question.

The Generating Hypothesis: (Peter Freyd - 1966) A map $f : X \to Y$ between finite CW-complexes is stably hull-homotopic if $\pi_*^s(f) = 0$.

This conjecture is not true unstably even for finite CW-complexes. Why?

This conjecture is also false for infinite dimensional complexes: A non-zero positive degree element in the mod-2 Steenrod algebra represents a *non-trivial* map $\phi : \Sigma^d H\mathbb{F}_2 \to H\mathbb{F}_2$. But $\pi_*^s(\phi)$ is clearly zero.

Only some partial (affirmative) results are known when the target is a sphere (Devinatz - 1990).

Nilpotence detection: When is a self map $f : \Sigma^d X \to X$ of a CW-complex stably nilpotent?? f is *stably nilpotent* if some iterate

$$\Sigma^{kt} X \to \Sigma^{(k-1)t} X \to \cdots \Sigma X \to X$$

of f is stably null-homotopic.

Theorem 3.1. *(Nishida -1973) A self-map $\Sigma^d S^n \to S^n$, for $d > 0$, is stably nilpotent!*

What about other finite CW-complexes? Start looking for generalised homology theories $E_*(-)$ which might detect stable nilpotence.

Theorem 3.2. *(Devinatz-Hopkins-Smith: 1988) There is a generalised homology theory* $MU_*(-)$ *(*Complex Bordism*) which detects stable nilpotence. More precisely, if* X *any finite CW-complex, a self map* $f : \Sigma^d X \to X$ *is stably nilpotent if and only if the endomorphism* $MU_*(f) : \Sigma^k MU_*(X) \to MU_*(X)$ *is nilpotent.*

This is truly remarkable theorem which has laid the foundation for much of modern homotopy theory. $MU_*(-)$ is often computable, so this theorem is quite powerful.

Existence of periodic maps: A map between CW-complexes is stably periodic if it is not stably nilpotent. Does every CW-complex admit a stable periodic self-map?

Why do we care about such maps? Such maps help us to detect new families in the stable homotopy groups of spheres!

Adams showed that for large n there is a stable periodic map $\Sigma^q M(p) \xrightarrow{A} M(p)$ $(q = 2(p-1))$ of the Moore space $(M(p) := S^n \bigcup_p e^{n+1})$ which induces isomorphism in K-theory - a generalised cohomology theory. This gives a "systematic family" in the stable homotopy groups of spheres - first constructed by Toda.

$$\begin{tikzcd}
\Sigma^{iq}M(p) \arrow[r, "\Sigma^{iq}A"] & \cdots \arrow[r] & \Sigma^{2q}M(p) \arrow[r, "\Sigma^{2q}A"] & \Sigma^{q}M(p) \arrow[r, "A"] & M(p) \arrow[d] \\
S^{n+iq} \arrow[u] \arrow[rrrr, dotted, "\alpha_i"] & & & & S^{n+1}
\end{tikzcd}$$

Note that α_i belongs to $\pi^s_{iq-1}(S^0)$ for all i.

Is the same true for any arbitrary finite CW-complex?

Morava K-theories: Fix a prime p. For $n \geq 0$, there are Morava K-theories which define generalised homology theories $K(n)_*(-)$. For example $K(0)_*(X) \cong H(X;\mathbb{Q})$, $K(\infty)_*(X) := H_*(X;\mathbb{F}_p)$.

A finite CW-complex is of type-n (at p) if $K(n)_i(X) = 0$ for $i < n$ and $K(n)_*(X) \neq 0$.

Theorem 3.3. *(Hopkins-Smith: 1998) Let X be a finite CW-complex of type $n > 0$ at prime p. Then there is a stable periodic self-map*

$$f : \Sigma^{d+i} X \to X \textit{ for some } i >> 0$$

such that $K(t)_* f$ is an isomorphism if $t = n$, and 0 for $t > n$.

So we have lots of periodic maps! — one of every type-n complex (Mitchell showed their existence in 1985). These will help us detect new families in $\pi_*^s(S)$.

4 Chromatic stable homotopy theory

If E is any generalised homology theory (e.g, Complex bordism, $K(n)_*$-theory), the E-local category of spaces is obtained by formally inverting the E-equivalences (maps that induces isomorphism on $E_*(-)$.) Such a category can be efficiently constructed using Bousfield localisation.

E-local homotopy category is, in general, much simpler that the stable homotopy category.

Chromatic view point: Understand the homotopy category by fracturing it into simpler pieces.

1. Understand the E-local homotopy category for various homology theories $E_*(-)$.
2. Assemble the E-local information from (1) to understand the original homotopy category.

For example, first study the $K(n)$-local homotopy categories for all n, and then patch the chromatic pieces.

Big Problem: Compute the homotopy groups of CW-complexes.

Step 1: Every CW-complex X is the union of all its finite dimensional sub-complexes.

$$X = \bigcup_{\alpha} X_{\alpha}.$$

Taking homotopy gives

$$\pi_*(X) = \operatorname*{colim}_{\alpha} \pi_*(X_{\alpha}).$$

Therefore restrict to finite dimensional CW-complexes!

Step 2: Computing the homotopy groups of finite dimensional CW-complexes is still a very hard problem. In fact:

There is no finite dimensional simply connected CW-complex for which all the homotopy groups are known!

However, one should not abandon all hope. The *stable* homotopy groups of finite dimensional CW-complexes are more tractable. So compute $\pi_*^s(X)$ for finite CW-complexes X.

Step 3: (Serre) If X is any finite dimensional CW-complex, then $\pi_t^s(X)$ is a finitely generated abelian group for all t. So try to understand the p-primary part of every p.

For every prime p, there is a finite p-local complex $X_{(p)}$ - i.e., a complex $X_{(p)}$ that satisfies

$$\pi_*^s(X_{(p)}) \cong \pi_*^s(X) \otimes \mathbb{Z}_{(p)}.$$

$\pi_*^s(X)$ can be recovered from the knowledge of $\pi_*^s(p)$ for various primes.

This is how people tried to compute the stable homotopy groups of spheres; for example one uses the classical Adams spectral sequence:

$$\mathrm{Ext}_{A_p}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_*^s(S^0_{(p)})$$

Step 4: Let X be a finite p-local complex. Now we can understand X by viewing it in the $E(n)$-local categories for all n. (In the $E(n)$-local category, we invert $K(i)_*$ equivalences for $0 \leq i \leq n$.)

There is a localisation functor L_n which when applied to X, puts it in the $E(n)$-local category. These localisations can be assembled into the so-called *Chromatic Tower* from which X can be recovered.

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & L_2(X) \arrow[d] \\
 & L_1(X) \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & L_0(X)
\end{tikzcd}$$

Figure 1: The Chromatic Tower of X

Theorem 4.1. *(Hopkins-Ravenel:1987)* ***Chromatic convergence theorem:*** *If X is a finite p-local complex, then*

$$X \simeq \underset{n}{holim}\, L_n X.$$

There are spectral sequences which converge to $\pi_*^s(L_n X)$ – Adams-Novikov spectral sequence or the Homotopy fixed point set spectral sequence (Hopkins-Miller).

5 Axiomatic stable homotopy theory

There are a bunch of axioms which define stable homotopy theory. These axioms are similar to the Eilenberg-Steenrod axioms which define singular homology theory.

One of the axioms which define stable homotopy theory is the following!

Finiteness axiom: The full subcategory of finite objects in $\mathcal{S}$ is equivalent to the full subcategory of the Spanier-Whitehead category consisting of finite dimensional CW-complexes.

One gets *generalised stable homotopy theories* by dropping this finiteness axiom – this is analogous to dropping the dimension axiom ($E_i(X) = 0$ for $i \neq 0$) from the Eilenberg-Steenrod axioms to obtain generalised homology theories.

Modern View point:(Hovey-Palmieri-Strickland: 1997) A *stable homotopy category* is a "sufficiently well-behaved" triangulated category - These are categories that are formally similar to the stable homotopy category category of spectra. E.g. The derived category of a ring, stable module category of a group algebra etc.

Homotopy Category of spaces	**Homotopy category of R**
Sphere (S)	Ring R
Suspension (Σ)	Shift $(\Sigma X)_n = X_{n-1}$
Wedge $(\vee)$	Direct sum $\oplus$
Smash product $(\wedge)$	Tensor product $\otimes_R$
Homotopy: $X \otimes I \to Y$	Chain Homotopy $X_\bullet \otimes_R I \to Y_\bullet$
Cofibre	Algebraic mapping cone
Homotopy groups $\pi_*(X)$	Homology groups $H_*(X)$
Weak Equivalences	Quasi-isomorphisms
Finite CW-complexes	Perfect complexes.

The derived category $D(R)$ of R is obtained from its homotopy category by inverting the Quasi-isomorphisms. This is an example of a stable homotopy category.

In $D(R)$, one can study all aforementioned Global problems of homotopy theory like the Generating hypothesis and nilpotence detection.

Theorem 5.1. *(Hopkins - 1985) Let* $f : X_\bullet \to Y_\bullet$ *be a self-map of a perfect complexes. Then* f *is tensor-nilpotent (i.e.,* $f^{\otimes n} = 0$ *for some integer* n*) if and only if*

$$f \otimes K(p) : X_\bullet \otimes K(p) \to Y_\bullet \otimes K(p)$$

is zero for all primes p*.*

$K(p)$ is the fraction field of the domain R/p - they play the role of the Morava K-theories.

There is a similar nilpotence theorem due to John Carlson for the stable module category.