Solution for Appendix B and §3.3

Appendix B:

Problem 25: Prove that the given function is injective.
(a) \( f : \mathbb{R} \to \mathbb{R}; f(x) = 2x \). Let \( a, b \in \mathbb{R} \) such that \( f(a) = f(b) \). Then \( 2a = 2b \Rightarrow a = b \), so \( f \) is injective.
(b) \( f : \mathbb{R} \to \mathbb{R}; f(x) = x^3 \). Let \( a, b \in \mathbb{R} \) such that \( f(a) = f(b) \). Then \( a^3 = b^3 \Rightarrow \sqrt[3]{a^3} = \sqrt[3]{b^3} \Rightarrow a = b \). Hence \( f \) is injective.
(c) \( f : \mathbb{Z} \to \mathbb{Q}; f(x) = (x/7) \). Let \( a, b \in \mathbb{Z} \) such that \( f(a) = f(b) \). Then \( a/7 = b/7 \Rightarrow a = b \). Hence \( f \) is injective.
(d) \( f : \mathbb{R} \to \mathbb{R}; f(x) = -3x + 5 \). Let \( a, b \in \mathbb{R} \) such that \( f(a) = f(b) \). Then \( -3a + 5 = -3b + 5 \Rightarrow -3a = -3b \Rightarrow a = b \). Hence \( f \) is injective.

Problem 26: Prove that the given function is surjective.
(a) \( f : \mathbb{R} \to \mathbb{R}; f(x) = x^3 \). Let \( b \in \mathbb{R} \). Then \( f(\sqrt[3]{b}) = (\sqrt[3]{b})^3 = b \), so \( f \) is surjective.
(b) \( f : \mathbb{Z} \to \mathbb{Z}; f(x) = x - 4 \). Let \( b \in \mathbb{Z} \). Then \( f(b + 4) = (b + 4) - 4 = b \), hence \( f \) is surjective.
(c) \( f : \mathbb{R} \to \mathbb{R}; f(x) = -3x + 5 \). Let \( b \in \mathbb{R} \). Then \( f(\frac{b-5}{3}) = -3(\frac{b-5}{3}) + 5 = b - 5 + 5 = b \). Hence \( f \) is surjective.
(d) \( f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}; f(a, b) = \begin{cases} a/b, & \text{if } b \neq 0 \\ 0, & \text{if } b = 0 \end{cases} \).

Let \( c \in \mathbb{Q} \). By definition of a rational number, there exist integers \( a, b \) such that \( b \neq 0 \) and \( c = a/b \). Therefore, \( f(a, b) = a/b = c \) and hence \( f \) is surjective.

Problem 27: Let \( f : B \to C \) and \( g : C \to D \) be functions. Prove:
(a) If \( f \) and \( g \) are injective, then \( g \circ f \) is injective.
(b) If \( f \) and \( g \) are surjective, then \( g \circ f \) is surjective.

(a) Assume \( f \) and \( g \) are injective and let \( a, b \in B \) such that \( g \circ f(a) = g \circ f(b) \). Then \( g(f(a)) = g(f(b)) \Rightarrow f(a) = f(b) \) since \( g \) is injective. But \( f(a) = f(b) \Rightarrow a = b \) since \( f \) is injective. Therefore, \( g \circ f \) is injective.

(b) Assume \( f \) and \( g \) are surjective. Let \( d \in D \). Then, since \( g \) is surjective, there exists \( a \in C \) such that \( g(c) = d \). Also, since \( f \) is surjective, there exists \( b \in B \) such that \( f(b) = c \). Hence \( g \circ f(b) = g(f(b)) = g(c) = d \), so \( g \circ f \) is surjective.

Q.E.D.
Section 3.3

**Problem 17:** Show that \( S = \{0, 4, 8, 12, 16, 20, 24\} \) is a subring of \( \mathbb{Z}_{28} \). Then prove that the map \( f : \mathbb{Z}_7 \to S \) given by \( f([x]_7) = [8x]_{28} \) is an isomorphism.

Note that we can characterize the set \( S \) as \( S = \{[4q]_{28} : q \in \mathbb{Z}\} \). By inspection we see that \([0]_{28} \in S\). Also, for any \( a, b \in S \) we have \( a = [4q]_{28} \) and \( b = [4r]_{28} \) for some integers \( q, r \in \mathbb{Z} \). So \( a + b = [4q] + [4r] = [4(q + r)] \in S \), \( ab = [4q][4r] = [4(4qr)] \in S \), and \( -a = [-4q] = [4(-q)] \in S \). By Theorem 3.2, \( S \) is a subring of \( \mathbb{Z}_{28} \).

Next, we show \( f \) is an isomorphism by showing Properties (H1) and (H2) hold and by showing \( f \) is one-to-one and onto. Let \( m, n \in \mathbb{Z}_7 \). Then

\[
f(m + n) = [8(m + n)]_{28} = [8m]_{28} + [8n]_{28} = f(m) + f(n)
\]

and

\[
f(m)f(n) = [8m]_{28}[8n]_{28} = [64mn]_{28} = [8mn]_{28} = f(mn).
\]

So \( f \) is a homomorphism. Finally, we note that \( f(0) = 0, f(1) = 8, f(2) = 16, f(3) = 24, f(4) = 32 = 4, f(5) = 40 = 12, f(6) = 48 = 20 \). We note that every element of \( S \) appears as \( f \) of something and no element of the codomain is mentioned twice, so \( f \) is both one-to-one and onto. Hence \( f \) is an isomorphism.

Q.E.D.