November 2001 Course 1 Examination, Problem No. 25, also P Sample Exam Questions, Problem No. 111, and Dr. Ostaszewski’s online exercise posted January 16, 2010

Once a fire is reported to a fire insurance company, the company makes an initial estimate, $X$, of the amount it will pay to the claimant for the fire loss. When the claim is finally settled, the company pays an amount, $Y$, to the claimant. The company has determined that $X$ and $Y$ have the joint density function

$$f(x, y) = \frac{2}{x^2(x-1)} y^{\frac{2x-1}{x-1}}$$

for $x > 1$, $y > 1$. Given that the initial claim estimate by the company is 2, determine the probability that the final settlement amount is between 1 and 3.

A. $\frac{1}{9}$  B. $\frac{2}{9}$  C. $\frac{1}{3}$  D. $\frac{2}{3}$  E. $\frac{8}{9}$

Solution.

We will use subscripts in the densities to indicate their random variables. The probability that we are looking for is

$$\Pr \left( 1 < Y < 3 \mid X = 2 \right) = \int_1^3 f_Y(y \mid X = 2) dy = \int_1^3 \frac{f_{X,Y}(2,y)}{f_X(2)} dy.$$

We have

$$f_{X,Y}(2,y) = \frac{2}{4(2-1)} y^{-\frac{(4-1)}{2-1}} = \frac{1}{2} y^{-3},$$

for $y > 1$, and

$$f_X(2) = \int_{\text{all } y} f_{X,Y}(2,y) dy = \int_1^\infty \frac{1}{2} y^{-3} dy = -\frac{1}{4} y^{-2} \bigg|_{y=1}^{y=\infty} = \frac{1}{4}.$$

Therefore,

$$f_Y(y \mid X = 2) = \frac{f_{X,Y}(2,y)}{f_X(2)} = \frac{1}{2} y^{-3} = 2y^{-3}.$$
for $y > 1$. Finally

$$\Pr(1 < Y < 3 \mid X = 2) = \int_1^3 2y^{-3} dy = \left( -y^{-2} \right) \bigg|_{y=1}^{y=3} = \left( -\frac{1}{9} \right) - (-1) = \frac{8}{9}. $$

Answer E.