May 2000 Course 1 Examination, Problem No. 11, also P Sample Exam Questions, Problem No. 112, and Dr. Ostaszewski’s online exercise posted January 23, 2010

A company offers a basic life insurance policy to its employees, as well as a supplemental life insurance policy. To purchase the supplemental policy, an employee must first purchase the basic policy. Let $X$ denote the proportion of employees who purchase the basic policy, and $Y$ the proportion of employees who purchase the supplemental policy. Let $X$ and $Y$ have the joint density function $f(x, y) = 2(x + y)$ on the region where the density is positive. Given that 10% of the employees buy the basic policy, what is the probability that fewer than 5% buy the supplemental policy?

A. 0.010  B. 0.013  C. 0.108  D. 0.417  E. 0.500

Solution.
We must have $Y \leq X$, as no more than proportion $X$ buy the supplementary policy. The region of positive joint density is the triangular region $0 \leq y \leq x \leq 1$ (below the line $y = x$, inside the unit square). We want to find the conditional probability $\Pr(Y < 0.05|X = 0.1)$. The conditional density for $Y$ given $X = 0.1$ is

$$f_Y(y|X = 0.1) = \frac{f_{X,Y}(0.1, y)}{f_X(0.1)}, \text{ where } f_X(0.1) \text{ is the marginal density of } X \text{ at } 0.1.$$ We find it as (note that $0 \leq y \leq x = 0.10$ in the integral)

$$f_X(0.1) = \int_0^{0.1} f_{X,Y}(0.1, y) \, dy = \int_0^{0.1} 2(0.1 + y) \, dy = 0.03.$$ Therefore,

$$f_Y(y|X = 0.1) = \frac{f_{X,Y}(0.1, y)}{f_X(0.1)} = \frac{2(0.1 + y)}{0.03},$$

and

$$\Pr(Y < 0.05|X = 0.1) = \int_0^{0.05} \frac{2(0.1 + y)}{0.03} \, dy = \frac{20}{3} \cdot 0.05 + \left( \frac{100y^2}{3} \right)_{y=0.05} \approx 0.4167.$$ Answer D.