A company offers earthquake insurance. Annual premiums are modeled by an exponential random variable with mean 2. Annual claims are modeled by an exponential random variable with mean 1. Premiums and claims are independent. Let \( X \) denote the ratio of claims to premiums. What is the density function of \( X \)?

\begin{align*}
\text{A. } & \frac{1}{2x+1} \\
\text{B. } & \frac{2}{(2x+1)^2} \\
\text{C. } & e^{-x} \\
\text{D. } & 2e^{-2x} \\
\text{E. } & xe^{-x}
\end{align*}

Solution.

Consider the following transformation \( X = \frac{U}{V}, \ Y = V \). Then the inverse transformation is \( U = XY, V = Y \). We know that \( f_{U,V}(u,v) = \frac{1}{2} e^{-u} e^{-\frac{v}{2}} \) for \( u > 0, \ v > 0 \). It follows that

\[
f_{X,Y}(x,y) = f_{U,V}(u(x,y),v(x,y)) \left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \frac{1}{2} e^{-vx} e^{-\frac{y}{2}} \left| \det \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \right| = \frac{1}{2} ye^{-vx} e^{-\frac{y}{2}}
\]

for \( xy > 0 \) and \( y > 0 \), or just \( x > 0 \) and \( y > 0 \). Therefore,

\[
f_X(x) = \int_0^\infty \frac{1}{2} ye^{-vx} e^{-\frac{y}{2}} dy = \int_0^\infty \frac{1}{2} ye^{-\left(\frac{x+1}{2}\right)y} dy = \frac{1}{2} \left( \frac{1}{x+\frac{1}{2}} \right) e^{-\left(\frac{x+1}{2}\right)^2} dy = \frac{1}{2} \cdot \frac{1}{x+\frac{1}{2}} \cdot \frac{1}{x+\frac{1}{2}} = \frac{2}{(2x+1)^2}.
\]

Answer B.

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