Exercise for October 11, 2008

**Study Note P-09-08, Problem No. 131**

Let $N_1$ and $N_2$ represent the numbers of claims submitted to a life insurance company in April and May, respectively. The joint probability function of $N_1$ and $N_2$ is

$$f_{N_1, N_2}(n_1, n_2) = \begin{cases} \frac{3}{4} \left( \frac{1}{4} \right)^{n_1-1} e^{-n_1} (1 - e^{-n_1})^{n_2-1} & \text{for } n_1 = 1, 2, 3 \ldots \text{and } n_2 = 1, 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected number of claims that will be submitted to the company in May if exactly 2 claims were submitted in April.

A. $\frac{3}{16} (e^2 - 1)$  
B. $\frac{3}{16} e^3$  
C. $\frac{3e}{4 - e}$  
D. $e^2 - 1$  
E. $e^2$

Solution.

We begin by finding the conditional probability function of $N_2$ given $N_1 = n_1$, i.e.,

$$f_{N_2 \mid N_1 = n_1} = \frac{f_{N_1, N_2}(n_1, n_2)}{f_{N_1}(n_1)}.$$

We have

$$f_{N_1}(n_1) = \sum_{n_2=1}^{\infty} f_{N_1, N_2}(n_1, n_2) = \frac{3}{4} \left( \frac{1}{4} \right)^{n_1-1} \sum_{n_2=1}^{\infty} e^{-n_1} (1 - e^{-n_1})^{n_2-1} = \frac{3}{4} \left( \frac{1}{4} \right)^{n_1-1} \cdot \frac{1}{1 - (1 - e^{-n_1})} = \frac{3}{4} \left( \frac{1}{4} \right)^{n_1-1} \cdot \frac{e^{-n_1}}{1 - (1 - e^{-n_1})} = \frac{3}{4} \left( \frac{1}{4} \right)^{n_1-1} \cdot \frac{e^{-n_1}}{1 - (1 - e^{-n_1})} = \frac{3}{4} \left( \frac{1}{4} \right)^{n_1-1}.$$

Therefore

$$f_{N_2 \mid N_1 = n_1} = \frac{f_{N_1, N_2}(n_1, n_2)}{f_{N_1}(n_1)} = e^{-n_1} (1 - e^{-n_1})^{n_2-1}.$$
This is actually the probability function of a geometric random variable with parameter 
\[ p = e^{-n_1} \]. The mean of this distribution is 
\[ \frac{1}{p} = \frac{1}{e^{-n_1}} = e^{n_1} \], and that equals \( e^2 \) when \( n_1 = 2 \).

Answer E.