May 2003 Course 1 Examination, Problem No. 39, also P Sample Exam Questions, Problem No. 95, and Dr. Ostaszewski’s online exercise posted October 3, 2009

X and Y are independent random variables with common moment generating function

\[ M(t) = e^{t^2/2}. \]

Let \( W = X + Y \) and \( Z = Y - X \). Determine the joint moment generating function \( M_{W,Z}(t_1,t_2) \) of W and Z.

A. \( e^{2t_1^2 + 2t_2^2} \)  
B. \( e^{(t_1-t_2)^2} \)  
C. \( e^{(t_1+t_2)^2} \)  
D. \( e^{2t_1t_2} \)  
E. \( e^{t_1^2 + t_2^2} \)

Solution.

We calculate directly from the definition

\[
M_{W,Z}(t_1,t_2) = E\left(e^{t_1W + t_2Z}\right) = E\left(e^{t_1(X+Y)+t_2(Y-X)}\right) = E\left(e^{(t_1-t_2)X}e^{(t_1+t_2)Y}\right) =
\]

\[
= E\left(e^{(t_1-t_2)X}\right) \cdot E\left(e^{(t_1+t_2)Y}\right) = e^{\frac{1}{2}(t_1-t_2)^2} e^{\frac{1}{2}(t_1+t_2)^2} = e^{\frac{1}{2}(t_1^2 - 2t_1t_2 + t_2^2) + \frac{1}{2}(t_1^2 + 2t_1t_2 + t_2^2)} = e^{t_1^2 + t_2^2}.
\]

Answer E.

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