Some additional notes on pension funding

Salary scale function
We use a function $s_x$ to describe salary growth in calculations of values of pension benefits and pension contributions, when those are related to a worker's salary. That function is also called a salary scale and its role is to project a salary from now into the future, especially near the time of retirement, because the final salary, or average of salaries several final years of work, are commonly used for calculation of pension benefits. But the salary scale can also be used to figure out past salary based on current salary. The salary scale function works this way

$$s_x = \frac{\text{Salary received at age } x, \text{ i.e., between exact age } x \text{ and exact age } x + 1}{s_y}. \frac{\text{Salary received at age } y, \text{ i.e., between exact age } x \text{ and exact age } y + 1}{s_y}.$$

We also consider the derivative of the salary scale function and call it salary rate. However, if the salary scale function uses only integers, i.e., salary does not change between two consecutive exact ages, then we just consider a difference quotient for an increment of 1, i.e.,

$$\text{Salary earned in age interval } (t, t + h)$$

for $h = 1$. In order to use the salary scale function to project salaries, we simply take

$$s_x \cdot \text{Salary received at age } x, \text{ i.e., between exact age } x \text{ and exact age } x + 1 =$$

$${s_x} \cdot \text{Salary received at age } y, \text{ i.e., between exact age } x \text{ and exact age } y + 1.$$

Pension plans
A defined benefit pension plan is a promise by the plan sponsor (typically an employer) to plan participants of some pension benefit (typically based on past salaries, years of service, and possibly age). A defined contribution pension plan is an arrangement whereby plan sponsor (also typically an employer) sets up an investment account for each plan participant, where either or both of these two parties can contribute funds to be invested and later on used to pay for the cost of retirement of the plan participant.

Note that a defined benefit plan promises a certain benefit, so it defines a benefit for a plan participant, and the plan participant often does not know how much money the plan sponsor actually contributes to fulfill that promise. A defined contribution pension plan, on the other hand, states (defines) a contribution to be made by plan sponsor (and possibly also a plan participant), but does not make any promise of any specific retirement benefit.

If a plan participant is required to make contributions to a pension plan, the plan is called contributory. In the United States, a qualified (i.e., accepted by the regulators and the tax agency as legitimate and approved for tax exemption) defined benefit plan must be non-contributory. But in Canada, there are defined benefit pension plans approved by regulators and tax authorities that are contributory. On the other hand, most defined
contribution pension plans in the United States are contributory. The percentage of pay paid into a defined contribution plan by the employer is called the *contribution rate*.

The *replacement ratio* of a pension plan is defined as the ratio of the pension income in the year after retirement to the final year (last year of work) salary.

In a defined benefit pension plan, there is always a formula for the amount of benefit. If it is only based on the final salary, the plan is called a *final salary plan*. If the formula is based on the salary throughout the years of employment, the plan is called a *career average salary plan*.

Because defined contribution pension plans do not establish the amount of benefit, it becomes important to calculate the amount of contribution required to achieve a desired replacement ratio.

**Exercise 1.**

An employer establishes a defined contribution (DC) plan. The contribution rate of the employee is 5% of salary, and the employer contributes 10% of salary. The employee joins the plan at age 25, and the retirement age is 65. Initial salary of the employee is $40,000 per annum, and pension plan contributions are made at the beginning of each year. Salary increases at 2% per year. The plan is invested in a portfolio that earns 7% annual effective. Upon retirement, accumulated funds are used to buy a life annuity due, paid monthly, and you are given that \( \dd{12}{65} = 11.25 \). Find the replacement ratio of this plan.

**Solution.**

Note that the last age of contributions is 64, because the employee retires at age 65. Let

\[
j = \frac{1.07}{1.02} - 1.
\]

The accumulated value of the contributions is

\[
40,000 \cdot 0.15 \cdot 1.07^{40} + 40,000 \cdot 1.02 \cdot 0.15 \cdot 1.07^{39} + 40,000 \cdot 1.02^2 \cdot 0.15 \cdot 1.07^{38} + \\
+ ... + 40,000 \cdot 1.02^j \cdot 0.15 \cdot 1.07 = \\
= 40,000 \cdot 0.15 \left( 1.02^0 \cdot 1.07^{40} + 1.02^1 \cdot 1.07^{39} + ... + 1.02^{39} \cdot 1.07 \right) = \\
= 40,000 \cdot 0.15 \cdot 1.07^{40} \cdot \frac{1}{1 + j} \left( 1 + \frac{1}{1 + j} + \frac{1}{(1 + j)^2} + ... + \frac{1}{(1 + j)^{39}} \right) = \\
= 40,000 \cdot 0.15 \cdot 1.07^{40} \cdot \dd{40}{40j} \approx 1,639,208.09.
\]

This purchases an annual benefit of

\[
\frac{1,639,208.09}{11.25} = 145,707.39.
\]

The final salary is projected to be

\[40,000 \cdot 1.02^{39} \approx 86,589.79\]
and the replacement ratio is
\[
\frac{145,707.39}{86,589.79} \approx 1.6827 = 168.27\%.
\]