P Sample Exam Questions, Problem No. 129, also Dr. Ostaszewski’s online exercise posted November 1, 2008

The cumulative distribution function for health care costs experienced by a policyholder is modeled by the function

\[ F_X(x) = \begin{cases} 1 - e^{-\frac{x}{100}} & \text{for } x > 0, \\ 0 & \text{otherwise}. \end{cases} \]

The policy has a deductible of 20. An insurer reimburses the policyholder for 100% of health care costs between 20 and 120 less the deductible. Health care costs above 120 are reimbursed at 50%. Let \( G \) be the cumulative distribution function of reimbursements given that the reimbursement is positive. Calculate \( G(115) \).

A. 0.683 B. 0.727 C. 0.741 D. 0.757 E. 0.777

Solution.

Let \( W \) be the unconditional reimbursement amount, and let \( Y \) be the reimbursement, given that the reimbursement is positive. We have

\[ W = \begin{cases} 0, & X \leq 20, \\ X - 20, & 20 < X \leq 120, \\ (120 - 20) + 0.5 \cdot (X - 120) = 40 + 0.5X, & X > 120. \end{cases} \]

But \( Y = (W | X > 20) \), so that

\[ Y = \begin{cases} X - 20, & X \leq 120, \text{ given that } X > 20, \\ 40 + 0.5X, & X > 120, \text{ given that } X > 20. \end{cases} \]

Note that when \( X \leq 120, \ X - 20 \leq 100 < 115. \) Also, \( X \) has exponential distribution, with memoryless property. We conclude that
\[ G(115) = \Pr(Y \leq 115) = \]
\[ = \Pr\left(\{X \leq 120|X > 20\} \cup \{40 + 0.5X \leq 115\} \cap \{X > 120|X > 20\}\right) = \]
\[ = \frac{\Pr(20 < X \leq 120)}{\Pr(X > 20)} + \Pr\left(\{120 < X \leq 150|X > 20\}\right) = \]
\[ = \frac{\Pr(20 < X \leq 120)}{\Pr(X > 20)} + \frac{\Pr(120 < X \leq 150)}{\Pr(X > 20)} = \frac{\Pr(20 < X \leq 150)}{\Pr(X > 20)} \]
\[ = \Pr(X \leq 130) = F_X(130) = 1 - e^{\frac{130}{100}} \approx 0.727. \]

Answer B.