A company agrees to accept the highest of four sealed bids on a property. The four bids are regarded as four independent random variables with common cumulative distribution function \( F(x) = \frac{1}{2}(1 + \sin \pi x) \) for \( \frac{3}{2} \leq x \leq \frac{5}{2} \). Which of the following represents the expected value of the accepted bid?

A. \( \pi \int_{\frac{3}{2}}^{\frac{5}{2}} x \cos \pi x \, dx \)

B. \( \frac{1}{16} \int_{\frac{3}{2}}^{\frac{5}{2}} (1 + \sin \pi x)^4 \, dx \)

C. \( \frac{1}{16} \int_{\frac{3}{2}}^{\frac{5}{2}} x(1 + \sin \pi x)^4 \, dx \)

D. \( \frac{1}{4} \pi \int_{\frac{3}{2}}^{\frac{5}{2}} \cos \pi x(1 + \sin \pi x)^3 \, dx \)

E. \( \frac{1}{4} \pi \int_{\frac{3}{2}}^{\frac{5}{2}} x \cos \pi x(1 + \sin \pi x)^3 \, dx \)

Solution.
Let \( X_1, X_2, X_3 \) and \( X_4 \) denote the four independent bids with common distribution function \( F \). Then if we define

\[ Y = \max(X_1, X_2, X_3, X_4), \]

this random variable represents the highest bid, and the cumulative distribution function \( F_Y \) of \( Y \) is

\[ F_Y(y) = \Pr(Y \leq y) = \Pr(\{X_1 \leq y\} \cap \{X_2 \leq y\} \cap \{X_3 \leq y\} \cap \{X_4 \leq y\}) = \Pr(X_1 \leq y)\Pr(X_2 \leq y)\Pr(X_3 \leq y)\Pr(X_4 \leq y) = (F(y))^4 = \frac{1}{16}(1 + \sin \pi y)^4, \]

for \( \frac{3}{2} \leq y \leq \frac{5}{2} \). It then follows that the density function \( g \) of \( Y \) is given by

\[ f_Y(y) = F_Y'(y) = \frac{1}{4}(1 + \sin \pi y)^3 \cdot \pi \cos \pi y = \frac{\pi}{4} \cos \pi y(1 + \sin \pi y)^3, \]
for $\frac{3}{2} \leq y \leq \frac{5}{2}$. Finally,

$$E(Y) = \int_{\frac{3}{2}}^{\frac{5}{2}} yf_y(y)dy = \frac{\pi}{4} y \cos \pi y (1 + \sin \pi y)^3 dy.$$ 

Answer E.