A random variable \( X \) has the cumulative distribution function
\[
F_X(x) = \begin{cases} 
0, & \text{for } x < 1, \\
\frac{x^2 - 2x + 2}{2}, & \text{for } 1 \leq x < 2, \\
1, & \text{for } x \geq 2. 
\end{cases}
\]

Calculate the variance of \( X \).

\[
\begin{array}{c}
A. \frac{7}{72} \\
B. \frac{1}{8} \\
C. \frac{5}{36} \\
D. \frac{4}{3} \\
E. \frac{23}{12}
\end{array}
\]

Solution.
It is crucial to realize that this is a mixed distribution with a point-mass at 1. This can be seen by analyzing the limit of CDF at 1 from the left (equal to 0) and the right-hand side limit of CDF at 1 (equal to 0.5), or, better yet, by drawing the graph of the CDF:

For a continuous distribution, the graph of its CDF is continuous non-decreasing curve. For a discrete distribution, the graph of its CDF consists of a series of horizontal lines, with jumps between them. If you see both jumps and pieces of continuous increasing curves, you are looking at a CDF of a mixed distribution. The jumps are at the points
where the discrete portion has positive probability function, and the sum of the jumps is the weight assigned to the discrete part of the mixed distribution. In this problem, there is only one jump: at 1. This means that the discrete portion of the mixed distribution is degenerate at 1. The weight of the discrete portion is the size of the only jump: \( \frac{1}{2} \). Given that, we can see that the distribution in this problem is a mixture of a degenerate distribution concentrated at 1, and a continuous distribution spread out between 1 and 2.

We decompose the CDF into a weighted-average of the CDF’s of those two distributions

\[
F_x(x) = \frac{1}{2} F_{x_1}(x) + \frac{1}{2} F_{x_2}(x).
\]

In this decomposition:

\[
F_{x_i}(x) = \begin{cases} 
0, & x < 1, \\
1, & x \geq 1,
\end{cases}
\]

as this is the formula for the CDF of a degenerate distribution concentrated at 1. The formula for \( F_{x_2} \) is found from simple algebra:

\[
F_{x_2}(x) = 2 \left( F_x(x) - \frac{1}{2} F_{x_1}(x) \right),
\]

resulting in

\[
F_{x_2}(x) = \begin{cases} 
0, & x < 1, \\
x^2 - 2x + 1, & 1 \leq x < 2, \\
1, & x \geq 2.
\end{cases}
\]

We also have

\[
f_{x_1}(x) = \begin{cases} 
1, & x = 1, \\
0, & \text{otherwise},
\end{cases}
\]

\[
f_{x_2}(x) = \begin{cases} 
2x - 2, & 1 < x < 2, \\
0, & \text{otherwise}.
\end{cases}
\]

Now we can calculate

\[
E(X_1) = E\left( (X_1)^2 \right) = 1,
\]

\[
E(X_2) = \frac{2}{3} \int x \cdot (2x - 2) dx = \frac{5}{3},
\]

\[
E\left( (X_2)^2 \right) = \frac{2}{3} \int x^2 \cdot (2x - 2) dx = \frac{17}{6},
\]

and finally

\[
\text{Var}(X) = E(X^2) - (E(X))^2 = \left( \frac{1}{2} E\left( (X_1)^2 \right) + \frac{1}{2} E\left( (X_2)^2 \right) \right) - \left( \frac{1}{2} E(X_1) + \frac{1}{2} E(X_2) \right)^2 =
\]

\[
= \left( \frac{1}{2} \cdot \frac{17}{6} \right) - \left( \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{5}{3} \right)^2 = \frac{23}{12} - \left( \frac{4}{3} \right)^2 = \frac{5}{36}.
\]

Answer C.

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