Exercise for February 24, 2007

Casualty Actuarial Society/Society of Actuaries Sample Course 1 Examination posted in August 1999, Problem No. 15

An insurance company issues insurance contracts to two classes of independent lives, as shown below.

<table>
<thead>
<tr>
<th>Class</th>
<th>Probability of Death</th>
<th>Benefit Amount</th>
<th>Number in Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.01</td>
<td>200</td>
<td>500</td>
</tr>
<tr>
<td>B</td>
<td>0.05</td>
<td>100</td>
<td>300</td>
</tr>
</tbody>
</table>

The company wants to collect an amount, in total, equal to the 95-th percentile of the distribution of total claims. The company will collect an amount from each life insured that is proportional to that life’s expected claim. That is, the amount for life $j$ with expected claim $E(X_j)$ would be $kE(X_j)$. Calculate $k$.

A. 1.30  B. 1.32  C. 1.34  D. 1.36  E. 1.38

Solution.

Let $X$ be a random claim for an insured individual from class A. Then $X = 200$ with probability $0.01$ and $X = 0$ with probability $0.99$. This gives $E(X) = 200 \cdot 0.01 = 2$ and $\text{Var}(X) = 200^2 \cdot 0.01 \cdot 0.99 = 396$.

Similarly, let $Y$ be a random claim for an insured individual from class B. Then $Y = 100$ with probability $0.05$ and $Y = 0$ with probability $0.95$. This gives $E(Y) = 100 \cdot 0.05 = 5$ and $\text{Var}(Y) = 100^2 \cdot 0.05 \cdot 0.95 = 475$.

For class A, total claims are $X_1 + X_2 + \ldots + X_{500}$, where $X_1,\ldots,X_{500}$ is a random sample from the distribution of $X$. Thus these total claims can be approximated with a normal distribution. For class B, total claims are $Y_1 + Y_2 + \ldots + Y_{300}$, where $Y_1,\ldots,Y_{300}$ is a random sample from the distribution of $Y$. These total claims can also be approximated by a normal distribution. The sum of these two totals is the amount of total claims, and because this is a sum of two independent random variables each of which can be approximated with a normal random variable, the sum itself can be approximated by a normal random variable. Let us write $T$ for that sum. Then
\( E(T) = 500 \cdot 2 + 300 \cdot 5 = 2500, \)

and
\[ \text{Var}(T) = 500 \cdot 396 + 300 \cdot 475 = 340500. \]
The company wants to collect the total amount of premiums \( t \) such that (note that \( Z \), as always, denotes a standard normal random variable)
\[ 0.95 = \Pr(T < t) = \Pr\left( \frac{T - E(T)}{\sqrt{\text{Var}(T)}} < \frac{t - E(T)}{\sqrt{\text{Var}(T)}} \right) \approx P \left( Z < \frac{t - E(T)}{\sqrt{\text{Var}(T)}} \right). \]
We must have
\[ 1.645 = \frac{t - E(T)}{\sqrt{\text{Var}(T)}}, \]
or
\[ t = 2500 + 1.645 \cdot \sqrt{340500} = 3459.89662. \]
Since \( k \cdot E(T) = t \), we conclude that proportionality constant \( k \) for the premium per insured should be
\[ k = \frac{t}{E(T)} = \frac{3459.89662}{2500} = 1.3840. \]

Answer E.