Exercise for March 3, 2007

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Suppose the remaining lifetimes of a husband and a wife are independent and uniformly distributed on the interval (0, 40). An insurance company offers two products to married couples:

- One which pays when the husband dies; and
- One which pays when both the husband and wife have died.

Calculate the covariance of the two payment times.

A. 0.0  B. 44.4  C. 66.7  D. 200.0  E. 466.7

Solution.

Let $H$ be the random time to death of the husband, $W$ be the time to death of the wife, and $X$ be the time to the second death of the two. Clearly, $X = \max(H, W)$. We have

$$f_H(h) = f_W(w) = \frac{1}{40}$$

for $0 \leq h \leq 40$, and $0 \leq w \leq 40$. Thus $E(H) = E(W) = 20$. Furthermore,

$$F_X(x) = \Pr(X \leq x) = \Pr(\max(H, W) \leq x) = \Pr(\{H \leq x\} \cap \{W \leq x\}) = \Pr(H \leq x) \cdot \Pr(W \leq x) = \frac{x}{40} \cdot \frac{x}{40} = \frac{x^2}{1600}.$$

This implies that $s_X(x) = 1 - \frac{x^2}{1600}$ for $0 \leq x \leq 40$, and

$$E(X) = \int_0^{40} \left(1 - \frac{x^2}{1600}\right) dx = 40 - \frac{40^3}{4800} = \frac{120}{3} = \frac{40}{3} = \frac{80}{3}.$$

In order to find covariance, we also need to find $E(XH) = E(\max(H, W))$. We separate the double integral into two parts: one based on the region where the wife lives longer and one based on the region where the husband lives longer, as illustrated in the graph below.
\[ E(h \max(h, w)) = \int \int h \max(h, w) \cdot f_h(h) \cdot f_w(w) \cdot dw dh = \]
\[ = \int_{h=0}^{h=40} \left( \int_{w=0}^{w=40} \left( \frac{h^2}{1600} \cdot \frac{1}{40} \cdot \frac{1}{40} \right) \right) dh + \int_{h=0}^{h=40} \left( \int_{w=0}^{w=40} \left( \frac{hw^2}{3200} \right) \right) dh = \]
\[ = \int_{h=0}^{h=40} \left( \frac{h^2}{1600} \right) dh + \int_{h=0}^{h=40} \left( \frac{hw^2}{3200} \right) dh = \]
\[ = \int_{h=0}^{h=40} \left( \frac{1}{2} h + \frac{1}{3200} h^3 \right) dh = \left( \frac{1}{4} h^2 + \frac{1}{12800} h^4 \right) \bigg|_{h=0}^{h=40} = \frac{1}{4} \cdot 40^2 + \frac{1}{12800} \cdot 40^4 = 600. \]

Finally,
\[ \text{Cov}(X, H) = E(XH) - E(X)E(H) = 600 - 20 \cdot \frac{80}{3} = 66 \frac{2}{3}. \]

Answer C.

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