Dr. Ostaszewski’s online exercise posted April 16, 2011

You are given that the joint probability density function of two random variables $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} 6x^2y & \text{for } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\Pr\left(\left|Y - X\right| < \frac{1}{2}\right)$.

A. 0.25  B. 0.33  C. 0.50  D. 0.66  E. 0.88

Solution.

We have

$$\Pr\left(\left|Y - X\right| < \frac{1}{2}\right) = \Pr\left(-\frac{1}{2} < Y - X < \frac{1}{2}\right) = \Pr\left(\left\{Y > X - \frac{1}{2}\right\} \cap \left\{Y < X + \frac{1}{2}\right\}\right) = \Pr\left(X - \frac{1}{2} < Y < X + \frac{1}{2}\right).$$

As the density is only positive inside the unit square, the probability sought can be calculated as the integral of the joint density over the region indicated in the figure below:
The shape of the region where \( \left\{ X - \frac{1}{2} < Y < X + \frac{1}{2} \right\} \) is somewhat irregular, and it is easier to calculate the probability of its complement, which is the integral over the two unmarked triangles:

\[
\Pr\left( \left\{ X - \frac{1}{2} < Y < X + \frac{1}{2} \right\}^c \right) = \int_0^{1/2} \int_{x-1/2}^{1} f_{X,Y}(x,y) \, dy \, dx + \int_{1/2}^{1} \int_{1/2}^{1} f_{X,Y}(x,y) \, dy \, dx =
\]

\[
= \frac{1}{2} \int_0^{1/2} \int_{x-1/2}^{1} 6x^2y \, dy \, dx + \frac{1}{2} \int_{1/2}^{1} \int_{1/2}^{1} 6x^2y \, dy \, dx = \frac{1}{2} \int_0^{1/2} 3x^2y^3 \bigg|_{y=1/2}^{y=1} \, dx + \frac{1}{2} \int_{1/2}^{1} 3x^2y^3 \bigg|_{y=1/2}^{y=0} \, dx =
\]

\[
= \frac{1}{2} \int_0^{1/2} 3x^2 \left( 1 - \left( x + \frac{1}{2} \right)^2 \right) \, dx + \frac{1}{2} \int_{1/2}^{1} 3x^2 \left( \left( x - \frac{1}{2} \right)^2 - 0 \right) \, dx.
\]

Now we note that

\[
1 - \left( x + \frac{1}{2} \right)^2 = \left( 1 - x - \frac{1}{2} \right) \left( 1 + x + \frac{1}{2} \right) = \left( \frac{1}{2} - x \right) \left( \frac{3}{2} + x \right) = \frac{3}{4} - x - x^2,
\]

and

\[
\left( x - \frac{1}{2} \right)^2 - 0 = \left( x - \frac{1}{2} \right)^2 = x^2 - x + \frac{1}{4},
\]

so that
We conclude that

\[
\Pr\left( X - \frac{1}{2} < Y < X + \frac{1}{2} \right) = 1 - \Pr\left( X - \frac{1}{2} < Y < X + \frac{1}{2} \right)^c = 1 - \frac{1}{8} = \frac{7}{8} = 0.875.
\]

Answer E.