Exercise for April 16, 2005

You are given that \( X \) and \( Y \) both have the same uniform distribution on \( [0, 1] \), and are independent. \( U = X + Y \) and \( V = \frac{X}{X+Y} \). Find the joint probability density function of \((U, V)\) evaluated at the point \( \left( \frac{1}{2}, \frac{1}{2} \right) \).

A. 0  
B. \( \frac{1}{4} \)  
C. \( \frac{1}{3} \)  
D. \( \frac{1}{2} \)  
E. 1

Solution.

We have \( X = UV \), and \( Y = U - UV \). Therefore,

\[
\frac{\partial (x, y)}{\partial (u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix} = -uv - u + uv = u.
\]

This implies that

\[
f_{U,V}(u,v) = f_{X,Y}(x(u,v),y(u,v)) \cdot \left| \frac{\partial (x,y)}{\partial (u,v)} \right| = 1 \cdot 1 \cdot |u| = u.
\]

This density at \( \left( \frac{1}{2}, \frac{1}{2} \right) \) equals \( \frac{1}{2} \).

Answer D.