The stock prices of two companies at the end of any given year are modeled with random variables $X$ and $Y$ that follow a distribution with joint density function

$$f(x, y) = \begin{cases} 2x, & \text{for } 0 < x < 1, \ x < y < x + 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is the conditional variance of $Y$ given that $X = x$?

A. $\frac{1}{12}$  
B. $\frac{7}{6}$  
C. $x + \frac{1}{2}$  
D. $x^2 - \frac{1}{6}$  
E. $x^2 + x + \frac{1}{3}$

Solution.

Let $f_x(x)$ be the marginal density function of $X$. Then

$$f_x(x) = \int_{x}^{x+1} 2xy\,dy = 2x(y)|_{y=x+1}^{y=x} = 2x(x+1) - 2x^2 = 2x$$

for $0 < x < 1$, and $f_x(x) = 0$ otherwise. Consequently,

$$f_y|X=x(y) = \frac{f_{x,y}(x,y)}{f_x(x)} = \begin{cases} 1, & \text{if } x < y < x + 1, \\ 0, & \text{otherwise.} \end{cases}$$

We see that the random variable $(Y|X=x)$ is uniform on the interval $(x, x+1)$, and therefore its mean is $x + \frac{1}{2}$, and its variance is $\frac{1}{12}$. You could calculate those parameters using calculus, but … you should not. You should know key properties of the uniform distribution.

Answer A.

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