A study of automobile accidents produced the following data:

<table>
<thead>
<tr>
<th>Model year</th>
<th>Proportion of all vehicles</th>
<th>Probability of involvement in an accident</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997</td>
<td>0.16</td>
<td>0.05</td>
</tr>
<tr>
<td>1998</td>
<td>0.18</td>
<td>0.02</td>
</tr>
<tr>
<td>1999</td>
<td>0.20</td>
<td>0.03</td>
</tr>
<tr>
<td>Other</td>
<td>0.46</td>
<td>0.04</td>
</tr>
</tbody>
</table>

An automobile from one of the model years 1997, 1998, and 1999 was involved in an accident. Determine the probability that the model year of this automobile is 1997.

A. 0.22  B. 0.30  C. 0.33  D. 0.45  E. 0.50

Solution.
Let us start by labeling the events: $A$ is the event that a car is involved in an accident, $E_{97}$ is the event that car model year is 1997, $E_{98}$ is the event that car model year is 1998, $E_{99}$ is the event that car model year is 1999, and $E_{O}$ is the event that car model year is other than 1997, 1998 or 1999. We are given: $\Pr(E_{97}) = 0.16$, $\Pr(E_{98}) = 0.18$, $\Pr(E_{99}) = 0.20$, $\Pr(E_{O}) = 0.46$, $\Pr(A|E_{97}) = 0.05$, $\Pr(A|E_{98}) = 0.02$, $\Pr(A|E_{99}) = 0.03$, and $\Pr(A|E_{O}) = 0.04$. It might seem that this is a Bayes’ Theorem problem, but note that the accident happened to a 1997, 1998, or a 1999 model, but not to one of the other years’ model, so that not all of the pieces of the partition \{\(E_{97}, E_{98}, E_{99}, E_{O}\)\} are considered, and we can’t use Bayes. We are actually asked to find the conditional probability:

$$\Pr(E_{97}|A \cap (E_{97} \cup E_{98} \cup E_{99}))$$. By the definition of conditional probability,
\[
\Pr\left( E_{97} | A \cap \left( E_{97} \cup E_{98} \cup E_{99} \right) \right) = \frac{\Pr\left( E_{97} \cap \left( A \cap \left( E_{97} \cup E_{98} \cup E_{99} \right) \right) \right)}{\Pr\left( A \cap \left( E_{97} \cup E_{98} \cup E_{99} \right) \right)} =
\]

\[
= \frac{\Pr\left( A \cap E_{97} \cap \left( E_{97} \cup E_{98} \cup E_{99} \right) \right)}{\Pr\left( A \cap E_{97} \right) + \Pr\left( A \cap E_{98} \right) + \Pr\left( A \cap E_{99} \right)} = \frac{\Pr\left( A \cap E_{97} \right) \cdot \Pr\left( E_{97} \right)}{\Pr\left( A \cap E_{97} \right) \cdot \Pr\left( E_{97} \right) + \Pr\left( A \cap E_{98} \right) \cdot \Pr\left( E_{98} \right) + \Pr\left( A \cap E_{99} \right) \cdot \Pr\left( E_{99} \right)} = \frac{0.05 \cdot 0.16}{0.05 \cdot 0.16 + 0.02 \cdot 0.18 + 0.03 \cdot 0.20} = 0.008 = 0.4545.
\]

Of course the reasoning of this solution is exactly the same as the proof of the Bayes' Theorem, and we can in fact observe that if we consider all events conditional on the events
\[
E_{97}^C = E_{97} \cup E_{98} \cup E_{99},
\]

events
\[
B_{97} = E_{97} \cap E_{97}^C, \quad B_{98} = E_{98} \cap E_{97}^C, \quad \text{and} \quad B_{99} = E_{99} \cap E_{97}^C,
\]

form a partition of \( E_{97}^C = E_{97} \cup E_{98} \cup E_{99} \), and for \( i = 97, 98, \) or \( 99 \), by writing
\[
A^* = A \cap E_{97}^C,
\]

we get
\[
\Pr\left( A^* | B_i \right) = \frac{\Pr\left( A \cap E_{97}^C \cap E_i \cap E_{97}^C \right)}{\Pr\left( E_i \cap E_{97}^C \right)} = \frac{\Pr\left( A \cap E_i \right)}{\Pr\left( E_i \right)} = \Pr\left( A | E_i \right).
\]

Hence, if we treat \( E_{97}^C \) as a new probability space, the assumptions of the Bayes' Theorem are satisfied, and we can calculate the probability sought as
\[
\Pr\left( B_{97} | A^* \right) = \frac{\Pr\left( A^* | B_{97} \right) \cdot \Pr\left( B_{97} \right)}{\Pr\left( A^* | B_{97} \right) \cdot \Pr\left( B_{97} \right) + \Pr\left( A^* | B_{98} \right) \cdot \Pr\left( B_{98} \right) + \Pr\left( A^* | B_{99} \right) \cdot \Pr\left( B_{99} \right)} = \frac{\Pr\left( A \cap E_{97} \right) \cdot \Pr\left( E_{97} \right)}{\Pr\left( A \cap E_{97} \right) \cdot \Pr\left( E_{97} \right) + \Pr\left( A \cap E_{98} \right) \cdot \Pr\left( E_{98} \right) + \Pr\left( A \cap E_{99} \right) \cdot \Pr\left( E_{99} \right)} = \frac{0.05 \cdot 0.16}{0.05 \cdot 0.16 + 0.02 \cdot 0.18 + 0.03 \cdot 0.20} = 0.008 = 0.4545.
\]

We could also do this problem by drawing a probability tree, and using only a portion of it that corresponds to model years 1997, 1998 and 1999, excluding other years’ models:
Then

\[ Pr(1997 \text{ model} | \text{Accident}) = \]

\[ = \frac{0.05 \cdot 0.16}{0.05 \cdot 0.16 + 0.02 \cdot 0.18 + 0.03 \cdot 0.20} = \frac{0.008}{0.0176} \approx 0.4545. \]

Answer D.