A city has just added 100 new female recruits to its police force. The city will provide a pension to each new hire who remains with the force until retirement. In addition, if the new hire is married at the time of her retirement, a second pension will be provided for her husband. A consulting actuary makes the following assumptions:

(i) Each new recruit has a 0.4 probability of remaining with the police force until retirement.

(ii) Given that a new recruit reaches retirement with the police force, the probability that she is not married at the time of retirement is 0.25.

(iii) The number of pensions that the city will provide on behalf of each new hire is independent of the number of pensions it will provide on behalf of any other new hire.

Determine the probability that the city will provide at most 90 pensions to the 100 new hires and their husbands.

A. 0.60  B. 0.67  C. 0.75  D. 0.93  E. 0.99

Solution.

Let $X_1,\ldots,X_{100}$ denote the number of pensions that will be provided to each new recruit. Under the assumptions of the problem each of these variables has the distribution, for $i = 1,\ldots,100$,

$$
X_i = \begin{cases} 
0, & \text{with probability } 1 - 0.4 = 0.6, \\
1, & \text{with probability } 0.4 \cdot 0.25 = 0.1, \\
2, & \text{with probability } 0.4 \cdot 0.75 = 0.3. 
\end{cases}
$$

Therefore, $E(X_i) = 0 \cdot 0.6 + 1 \cdot 0.1 + 2 \cdot 0.3 = 0.7$, $E(X_i^2) = 0^2 \cdot 0.6 + 1^2 \cdot 0.1 + 2^2 \cdot 0.3 = 1.3$, and

$$
\text{Var}(X_i) = E(X_i^2) - (E(X_i))^2 = 1.3 - 0.7^2 = 0.81.
$$

Since $X_1,\ldots,X_{100}$ are assumed by the consulting actuary to be independent and identically distributed, the Central Limit Theorem applies to $S = X_1 + \ldots + X_{100}$, which is approximately normally distributed with mean
\[ E(S) = E(X_1) + \ldots + E(X_{100}) = 100 \cdot 0.7 = 70, \]
and variance
\[ \text{Var}(S) = \text{Var}(X_1) + \ldots + \text{Var}(X_{100}) = 100 \cdot 0.81 = 81. \]
Using continuity correction, with \( W \sim N(70, 9^2) \) and \( Z \sim N(0,1) \), we get
\[ \Pr(S \leq 90) \approx \Pr(W \leq 90.5) = \Pr\left( \frac{W - 70}{9} \leq \frac{90.5 - 70}{9} \right) = \Pr(Z \leq 2.28) = 0.9887. \]
Answer E.

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