Sample Course 1 Examination, Problem No. 23
The value, \( v \), of an appliance is based on the number of years since purchase, \( t \), as follows:
\[
v(t) = e^{7 - 0.2t}
\]
If the appliance fails within seven years of purchase, a warranty pays the owner the value of the appliance. After seven years, the warranty pays nothing. The time until failure of the appliance failing has an exponential distribution with mean 10. Calculate the expected payment from the warranty.

A. 98.70        B. 109.66        C. 270.43        D. 320.78        E. 352.16

Solution.
If \( T \) is the time to failure, then we are given that
\[
f_T(t) = 0.1e^{-0.1t}
\]
The expected payment is therefore
\[
\int_0^7 v(t)f_T(t) \, dt = \int_0^7 e^{7 - 0.2t} \cdot \frac{1}{10} e^{-0.1t} \, dt
\]
\[
= \frac{1}{10} \left( -1 \right) \left[ e^{7 - 0.3t} \right]_0^7 - \frac{1}{3} \left( e^{7 - 2.1} - e^7 \right)
\]
\[
= -\frac{1}{3} \cdot (134.29 - 1096.63) \approx 320.78.
\]
You can also use the Darth Vader Rule, but you have to apply it to the payment amount, not \( T \). We define this new random variable describing the payment amount:
\[
U = \begin{cases} 
  e^{7 - 0.2t}, & 0 < t \leq 7, \\
  0, & t > 7.
\end{cases}
\]
It is clear that \( U \) is nonnegative, as it is a value of the exponential function, or 0. We have
\[
\Pr(U = 0) = \Pr(T > 7) = e^{-7} = e^{-0.7}.
\]
Note also that the value function \( v(t) = e^{7 - 0.2t} \) is decreasing for \( 0 < t < 7 \), with its largest value being \( v(0) = e^7 \) and its smallest value being \( v(7) = e^{5.6} \). This means that
\[ s_U(u) = 1 - e^{-0.7} \text{ for } u \leq e^{5.6}. \] Recall that \( F_t(t) = \Pr(T \leq t) = 1 - e^{-0.1t} \) for \( t > 0. \) For \( u > e^{5.6} \) we have
\[
s_U(u) = \Pr(U \geq u) = \Pr(e^{7-0.2T} \geq u) = \Pr(7 - 0.2T \geq \ln u) = \Pr(T \leq 35 - 5 \ln u) = 1 - e^{-0.1(35 - 5 \ln u)} = 1 - e^{-3.5} \cdot e^{0.5 \ln u} = 1 - e^{-3.5} \cdot \sqrt{u}
\]
as long as \( u \leq e^7 \). and \( s_U(u) = 0 \) for \( u > e^7 \). Therefore
\[
E(U) = \int_0^{e^{5.6}} (1 - e^{-0.7}) \, du + \int_{e^{5.6}}^{e^7} (1 - e^{-3.5} \sqrt{u}) \, du =
\]
\[
= (e^{5.6} - e^{4.9}) + (e^7 - e^{5.6}) - \left( \frac{2}{3} e^{-3.5} \cdot u^{1.5} \bigg|_{u=e^7} \right) =
\]
\[
= (e^7 - e^{4.9}) - \frac{2}{3} (e^7 - e^{4.9}) = \frac{1}{3} e^7 - \frac{1}{3} e^{4.9} \approx 320.781126.
\]
Answer D.

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