1. Fundamental concepts of the theory of interest

If a unit amount is placed in an account, its accumulated value at time $t$ is denoted by $a(t)$ and called the accumulation function. It must be equal to 1 (amount invested) plus the interest earned over the period $[0, t]$. We assume generally that $a(t)$ is an increasing continuous function. If the amount invested is different than 1, we typically denote the accumulated value in the account at time $t$ by $A(t)$ and call it the amount function. The two functions are related by the formula $A(t) = A(0) \cdot a(t)$, which effectively states that every monetary unit invested earns interest at the same rate.

Assume at first that rate of return (interest rate) $i$ is constant in the period of time considered. We will use a year as a unit of time. If the interest is compounded (i.e., added to principal invested) once a year (i.e., annually), the rate of return $i$ is called the effective rate of return (over a year). If the compounding frequency is different than once a year, the annual rate is called a nominal rate of return and denoted $i^{(m)}$ where $m$ is the compounding frequency per year. The relationship between the two measurements of interest is:

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m.$$

If the interest rate is not constant, we define the effective annual interest rate as the interest earned over a given year divided by the amount invested at the year’s beginning. Thus the effective annual rate $i_t$ during the $t$-th year, i.e., in the time period $[t-1, t]$ is defined by:

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)} = \frac{A(t) - A(t-1)}{A(t-1)},$$

assuming that the only principal invested is either 1 for the function $a(t)$, or $A(0)$ for the function $A(t)$. We typically assume that the period of time over which the interest rate is defined is a year, but this is not in any way required, it is merely a convention. We should, however, note that the unit of time over which the interest rate is defined should be the same as the one used for actual counting of time.

If the effective rate of interest $i$ is constant every year and the interest earned is reinvested in the account, we call the resulting accumulation function the compound interest accumulation function. Its form is:

$$a(t) = (1 + i)^t,$$

i.e., is exponential. If, however, the interest earned every year does not earn additional interest, the resulting form of the accumulation function:

$$a(t) = 1 + ti$$

is termed the simple interest accumulation function. The two functions are equal for $t = 0$ or $t = 1$, and the compound interest function increases much more rapidly for $t > 1$. However, it should be noted that for $0 < t < 1$, simple interest actually exceeds compound interest.

Under compound interest, the expression $1 + i$ is called the accumulation factor, and it equals the accumulated value of a monetary unit after one year. Its reciprocal,
\[ v = \frac{1}{1 + i}, \] is called the \textit{discount factor}, and is the present value of a monetary unit paid a year from now. Note also that \( v' \), the \( t \)-year discount factor, is the present value of a monetary unit paid \( t \) years from now (\( t \) need not be an integer).

The \textit{effective annual rate of discount} in the year \( t \), \( \hat{d}_t \), is defined as the interest earned over a year divided by the accumulated amount at the year-end. In the case of the compound interest accumulation function, the effective rate of discount is constant and written as \( d \). We have then:

\[ d = \frac{a(t) - a(t-1)}{a(t)} = \frac{i}{1 + i}. \]

Note that

\begin{align*}
  v + d &= \frac{1}{1 + i} + \frac{i}{1 + i} = 1, \\
  v &= 1 - d, \quad d = 1 - v,
\end{align*}

and

\[ vi = \frac{1}{1 + i} = d. \]

It is worth remembering that if \( i = \frac{1}{n} \) for an integer \( n \), then

\[ d = \frac{1}{n + 1}. \]

The accumulation function produced by a constant rate of discount is:

\[ a(t) = (1 - d)^{-t} = \frac{1}{(1 - d)^t}. \]

Two measurements of interest are said to be \textit{equivalent} if for any amount of principal invested for any length of time they yield equal accumulated values at any time. It can be shown easily that two measurements of interest are equivalent if, and only if they produce the same accumulation function. The following identities hold for equivalent measurements of interest:

\[ 1 + i = \frac{1}{v} = \frac{1}{1 - d} = \left( 1 + \frac{i^{(m)}}{m} \right)^{m}. \]

For the nominal annual interest rate \( i^{(m)} \) compounded \( m \) times a year, the number \( m \) need not be an integer, the formula works equally well for an annual rate compounded, e.g., every eight months, with \( m = 1.5 \) in such a case.

Nominal rates of discount can also be considered. Constant nominal annual rate of discount compounded \( m \) times a year is denoted by \( d^{(m)} \), and the accumulation function produced by it is:

\[ a(t) = \left( 1 - \frac{d^{(m)}}{m} \right)^{-mt}, \]

so that the following holds for equivalent measurements of interest:
\[1 + i = \frac{1}{v} = \frac{1}{1 - d} = \left(1 + \frac{i^{(m)}}{m}\right)^m = \left(1 - \frac{d^{(r)}}{r}\right)^r\]

(we are using a different notation for the compounding frequency of nominal interest and nominal discount to indicate that the two need not be equal). We also have:

\[1 - \frac{d^{(m)}}{m} = v^m, \quad \frac{d^{(m)}}{m} = \frac{m}{i^{(m)}}, \quad \frac{1}{v^m} \cdot \frac{i^{(m)}}{m} = \frac{d^{(m)}}{m}.\]

As the compounding frequency increases for a deposit earning a fixed annual nominal interest rate \(i^{(m)}\), the amount of interest earned in a year increases, but only to a certain upper bound. The accumulated value at the end of one year will be:

\[
\lim_{m \to \infty} \left(1 + \frac{i^{(m)}}{m}\right)^m = \lim_{m \to \infty} \left(1 + \frac{1}{m} \frac{m}{i^{(m)}} \right)^{i^{(m)}} = e^{i^{(m)}}.
\]

We obtain the same kind of limit result when working with compound discount. The limiting nominal annual interest rate is said to be compounded continuously. It is commonly called the force of interest and denoted by \(\delta\). The accumulation function given by it is:

\[a(t) = e^{\delta t}.\]

We have the following identities for equivalent measurements of interest:

\[1 + i = \frac{1}{v} = \frac{1}{1 - d} = \left(1 + \frac{i^{(m)}}{m}\right)^m = \left(1 - \frac{d^{(r)}}{r}\right)^r = e^{\delta}.\]

\[i > i^{(2)} > i^{(3)} > \ldots > \delta > \ldots > d^{(3)} > d^{(2)} > d.\]

In general, the force of interest can vary over time and in such general case it is defined as:

\[\delta_t = \frac{a'(t)}{a(t)} = \frac{d}{dt} \ln(a(t)) = \frac{A'(t)}{A(t)}.\]

Note that:

\[A(t + \Delta t) - A(t) \approx A(t) \cdot \delta_t \cdot \Delta t.\]

Under varying force of interest:

\[a(t) = e^{\int_{\delta t} dt} = e^{\delta dt}.\]

In particular, if

\[\delta_t = c \frac{f'(t)}{f(t)} = c \frac{d}{dt} \ln(f(t))\]

for some function \(f(t)\) then
\[ a(t) = \left( \frac{f(t)}{f(0)} \right)^c. \]

Also, if we move money forward in time (accumulate) from time \( t_1 \) to time \( t_2 \), with \( t_2 > t_1 \), we multiply it by

\[ \frac{a(t_2)}{a(t_1)} = e^{\int_{t_1}^{t_2} \delta_s ds}, \]

while if we move money from time \( t_2 \) to time \( t_1 \) (i.e., discount it back in time) then we multiply it by

\[ \frac{a(t_1)}{a(t_2)} = e^{-\int_{t_1}^{t_2} \delta_s ds}. \]

Sometimes it is useful to know the rate of change, i.e., the derivative, of one measurement of interest with respect to another. This is particularly important when considering derivatives of financial assets prices with respect to the interest rate or the force of interest. We have, for example:

\[
\begin{align*}
\frac{d\delta}{di} &= \frac{d}{di} \ln(1 + i) = \frac{1}{1 + i} = v = 1 - d, \\
\frac{dv}{di} &= \frac{d}{di} \frac{1}{1 + i} = -\frac{1}{(1 + i)^2} = -v^2, \\
\frac{dv^i}{di} &= -tv^{i+1},
\end{align*}
\]

and

\[
\frac{dd}{di} = \frac{d(1 - v)}{di} = v^2 = (1 - d)^2.
\]

**May 2003 SOA/CAS Course 2 Examination, Problem No. 1**

Bruce deposits 100 into a bank account. His account is credited interest at a nominal rate of interest \( i \) convertible semiannually. At the same time, Peter deposits 100 into a separate account. Peter’s account is credited interest at a force of interest of \( \delta \). After 7.25 years, the value of each account is 200. Calculate \( i - \delta \).

A. 0.12%  
B. 0.23%  
C. 0.31%  
D. 0.39%  
E. 0.47%

**Solution.**

Bruce’s account after 7.25 years is worth:

\[
100 \left( 1 + \left( \frac{i^{(m)}}{m} \right)^m \right)^{m \cdot 7.25} = 100 \left( 1 + \frac{i}{2} \right)^{2 \cdot 7.25} = 100 \left( 1 + \frac{i}{2} \right)^{14.5} = 200.
\]

Therefore,

\[
1 + \frac{i}{2} = 2^{\frac{1}{14.5}}
\]
and \( i = 9.7929\% \). Peter’s account after 7.25 years is worth:

\[
100e^{7.25\delta} = 200.
\]

Thus, \( 7.25\delta = \ln 2 \) and \( \delta = \frac{\ln 2}{7.25} = 0.95607\% \). We conclude that

\[
(i - \delta) = 0.2322\%.
\]

Answer B.

**May 2003 Course 2 SOA/CAS Examination, Problem No. 12**

Eric deposits \( X \) into a savings account at time 0, which pays interest at a nominal rate of \( i \), compounded semiannually. Mike deposits \( 2X \) into a different savings account at time 0, which pays simple interest at an annual rate of \( i \). Eric and Mike earn the same amount of interest during the last 6 months of the 8-th year. Calculate \( i \).

A. 9.06%  
B. 9.26%  
C. 9.46%  
D. 9.66%  
E. 9.86%

**Solution.**

Eric’s interest in the last 6 months of the 8-th year is:

\[
X \left( 1 + \frac{i}{2} \right)^{7.5} \cdot \frac{i}{2} = X \left( 1 + \frac{i}{2} \right)^{15} \cdot \frac{i}{2}.
\]

Mike’s interest in the last 6 months of the 8-th year is \( 2X \cdot \frac{i}{2} \). Therefore

\[
X \left( 1 + \frac{i}{2} \right)^{15} \cdot \frac{i}{2} = 2X \cdot \frac{i}{2},
\]

so that \( \left( 1 + \frac{i}{2} \right)^{15} = 2 \), and \( i = 2 \left( \frac{1}{15} - 1 \right) \approx 9.46\% \).

Answer C.

**May 2003 SOA/CAS Course 2 Examination, Problem No. 50**

Jeff deposits 10 into a fund today and 20 fifteen years later. Interest is credited at a nominal discount rate of \( d \) compounded quarterly for the first 10 years, and at a nominal interest rate of 6% compounded semiannually thereafter. The accumulated balance in the fund at the end of 30 years is 100. Calculate \( d \).

A. 4.33%  
B. 4.33%  
C. 4.53%  
D. 4.63%  
E. 4.73%

**Solution.**

Writing the equation of value we have:

\[
10 \left( 1 - \frac{d}{4} \right)^{40} \cdot 1.03^{40} + 20 \cdot 1.03^{30} = 100.
\]

This implies

\[
\left( 1 - \frac{d}{4} \right)^{40} \approx 1.577,
\]
and consequently $1 - \frac{d}{4} = 0.98867052$, so that $d = 0.0453$.

Answer C.

2. Cash flows and interest measurement in a fund

In practice, deposits are made more frequently than just at the beginning. Withdrawals can be made, as well. Consider an account in which an amount of $C_0$ is initially invested, and at the end of the $k$-th year, additional amount of $\gamma_k$, $k = 1,\ldots,n$ is invested (the amounts could, in general, be positive or negative). Such a combination of payments will be termed a cash flow $C_0, \gamma_1,\ldots,\gamma_n$ of this account. Let $C_k$ be the account balance at the end of the $k$-th year. Then

$$C_k = C_{k-1} + iC_{k-1} + \gamma_k, k = 1,\ldots,n.$$  

After a simple rearrangement of the above formula we obtain:

$$C_n = (1 + i)^n C_0 + \sum_{k=1}^{n} (1 + i)^{n-k} \gamma_k.$$  

This formula basically tells us that the balance in the account at the end of the $n$-th year will be the accumulated value of the initial deposit plus the accumulated value of all additional deposits made up to the time $n$. Using the discount factor $v$, we can also rewrite as:

$$v^n C_n = C_0 + \sum_{k=1}^{n} v^k \gamma_k.$$  

Now assume that the payments made into an account are made continuously at a rate of $\gamma(t)$, a function of time $t$. This means that the payment amount in the infinitesimal period of time $(t,t + dt)$ is equal to $\gamma(t) dt$. Let $C(t)$ be the account value at time $t$. Assume also that the interest is compounded continuously with the force of interest equal to $\delta(t)$. This means that if at the time $t$ the account balance is $C(t)$, then after the infinitesimal period of time $(t,t + dt)$ the account value will increase by $C(t)\delta(t) dt$ (assuming $\delta(t)$ is positive, as if $\delta(t)$ is negative, this is a decrease). The total increase in the account value in the time period $(t,t + dt)$ is therefore equal to:

$$dC(t) = C(t)\delta(t) dt + \gamma(t) dt.$$  

In particular, it implies that the accumulated value at time $\tau > t$ of a payment $C(0)$ made at time 0, with continuous deposits at the rate $\gamma(t)$ (possibly negative), with continuous compounding at the rate $\delta(s)$ is:

$$C(\tau) = C(0)e^{\int_0^\tau \delta(s) ds} + \int_0^\tau \gamma(t)e^{\int_t^\tau \delta(s) ds} dt.$$  

In particular, the accumulated value of a single payment of $C(0)$ made at time 0 is:

$$C(\tau) = C(0)e^{\int_0^\tau \delta(s) ds}.$$
One problem that comes up when dealing with cash flows is the question of an interest rate that makes certain cash flows equivalent. For example, suppose that we have 30 monetary units paid every year at the end of the year for four years in return for 100 paid today. What is the interest rate that makes these two equal? It is called the internal rate of return, or the dollar-weighted rate of return. In order to find that interest rate \( i \), we set up the equation:

\[
100 = \frac{30}{1+i} + \frac{30}{(1+i)^2} + \frac{30}{(1+i)^3} + \frac{30}{(1+i)^4},
\]

or

\[
30v + 30v^2 + 30v^3 + 30v^4 - 100 = 0.
\]

If the payments are level, most financial calculators can find the interest rate. This can be also calculated using Solver function in Excel, and \( i \approx 7.713847\% \). The internal rate of return is also called the yield rate for the cash flow analyzed. It is important to note that finding the yield rate involves solving a polynomial equation, and because of that:
- There may be multiple solutions,
- There may be no real number solution,
- There may be solutions, which are complex numbers.

There are situations when a unique real solution exists: if all outflows occur before all inflows, or if the cash inflows can be treated as “deposits” and outflows as “withdrawals”, with the resulting account balance remaining positive throughout the period under consideration.

There is also a problem with the use of the internal rate of return (IRR) as a tool for evaluating a project in which the inflows are invested, and from which the outflows are returns – using IRR implies that cash flows received from the project are reinvested at that IRR, which may or may not (usually is not) be possible at the prevailing market rates.

In practical calculation of interest, it is sometimes necessary to determine a fractional time period of investment, when such time period is not an exact whole number of years. This is used particularly often with simple interest assumption. There are three common approaches in doing this:
1. Exact number of days of the investment, and 365 (or 366 in a leap year) days in a year.
2. Assume 30 days in a month and 360 days in a year. This is called “30/360” or ordinary simple interest.
3. Use the exact number of days and 360 days in a year. This is called the Banker’s Rule.

Before the age of computers and calculators, there was one special approximation used commonly that we will now present. If the cash flows are \( s_1, s_2, \ldots, s_n \) at times \( t_1, t_2, \ldots, t_n \), the standard problem calls for finding a time \( t \) such that the one sum payment

\[
s = \sum_{i=1}^{n} s_i \]

has the same present value as all cash flows, i.e.,

\[
sv' = \sum_{i=1}^{n} s_i v'^i.
\]

This can be solved for \( t \):
\[
t = -\frac{\ln\left(\sum_{i=1}^{n} \frac{S_i s_i}{s}\right)}{\delta}.
\]

A traditional, approximate method of equated time uses an approximation of \( t \), let us call it \( \bar{t} \). In this approximation:

\[
\bar{t} \approx \frac{\sum_{i=1}^{n} S_i s_i}{s}.
\]

This approximation results in \( t < \bar{t} \).

In general, when we value a set of cash flows, its value today is called the present value, while its value at some future time is termed the accumulated value. If the value is calculated at a time somewhere is between now and the future time horizon, it is called the current value.

Note that if two cash flows are equivalent at one point in time, they are equivalent at any other time. Also, a value of a linear combination of cash flows is a linear combination of values of individual cash flows. Finally, if a value \( V(t) \) is established at time \( t \) then the value \( V(s) \) at any other time may be calculated as:

\[
V(s) = v^{-s}V(t).
\]

One important consequence is also that you can compare cash flows, or calculate their values, at various points in time, and it is often convenient to pick a different reference date than just today, or a future horizon.

In the practice of insurance companies a problem of interest measurement of a fund over a year arises naturally. Let us introduce the following notation:

- \( A \) = initial fund balance, at the beginning of the year, time \( t = 0 \),
- \( B \) = final fund balance, at the end of the year, time \( t = 1 \),
- \( I \) = interest earned during the year, between \( t = 0 \) and \( t = 1 \),
- \( C_t \) = principal contributed (if positive) or withdrawn (if negative) at time \( t \),
- \( C = \sum_t C_t \) = total net contribution to the fund.

We have:

\[
B = A + C + I,
\]

\[
I = iA + \sum_{0 \leq t \leq 1} C_t \left( (1+i)^{1-t} - 1 \right).
\]

In order to find \( i \) we would have to solve the second equation for it, and the first equation would actually be the only source of the value of \( I \). This is not easy, and in practice one often uses a simple interest approximation:

\[
(1+i)^{1-t} - 1 \approx 1 + (1-t)i - 1 = (1-t)i.
\]

Therefore,

\[
i \approx \frac{I}{A + \sum_{0 \leq t \leq 1} C_t (1-t)} = \frac{I}{A \cdot 1 + \sum_{0 \leq t \leq 1} C_t (1-t)}.
\]
The numerator in this formula is the interest earned. The denominator of the formula for \( i \) so obtained is commonly called the exposure associated with \( i \). It represents the net amount-time exposed to earning interest: all amounts are multiplied by how long they stayed in the account during the year (\( A \) was there at the beginning of the year, so it had a full year, while each \( C_t \) was there for \( 1 - t \) years). The approximation of the internal rate of return obtained with this simple interest assumption is what is also commonly called the dollar-weighted rate of return, and this concept will mean such an approximation in these notes.

If the net principal contributions occur at time \( k \) on the average, i.e.,

\[
k = \sum_{0 \leq r \leq 1} t \cdot \frac{C_r}{C},
\]

then we use a simpler approximation:

\[
i \approx \frac{I}{A + C(1 - k)}.
\]

It would seem that these approximations provide a good practical measure of the return earned by a fund. But investment professional do not use dollar-weighted rate of return because it is very sensitive to the timing of cash flows, and such timing is often beyond their control, thus should not be used to measure their performance. Instead, the time-weighted rate of return, which is the return earned by a unit of a fund under consideration. If we write \( B_t \) for the account balance just before the contribution \( C_t \) is made, then the effective rate of return between two contributions \( C_s \) and \( C_t \) with \( s < t \), is \( j \) given by:

\[
1 + j = \frac{B_t}{B_s + C_s}.
\]

If the contributions are made at times \( t_1, t_2, \ldots, t_m \) then the time-weighted rate of return for the year is \( i \) such that:

\[
1 + i = (1 + j_1)(1 + j_2) \cdots (1 + j_m).
\]

Note that the amounts and timing of contributions do not affect the time-weighted rate of return.

One more practical problem encountered by insurance and investment companies, as well as trust divisions of banks, is allocation of interest to accounts. The nature of the problem is that assets of various customers are commingled, and sometimes they do not all get pro-rata share of the fund’s performance (this is the standard for mutual funds, but not for insurance companies, stable value funds, or commingled trust accounts). There are two standard methods of dealing with this problem:

- **Portfolio Method**: Each account is credited with an average rate based on earnings of the entire fund for the period. The drawback of this method is that when interest rates are rising, there is a disincentive for new deposits, as fixed coupon bonds in the portfolio bought with previous deposits are now declining in value, and this negative performance contribution is received by new money, as well.

- **Investment Year Method**: “New money” (i.e., new deposits) receive a different rate than “old money,” this rate is based on current market conditions. However, once investments stay in the account for a specified period of time (for insurance
companies offering fixed annuities, this is typically five to seven years, or the period during which a surrender charge is assessed), they eventually are credited the portfolio rate, just as the “old money”. The transition period from the “new money rate” to the portfolio rate can be handled either through a \textit{declining index system} (the rate is gradually changed from the “new money” rate to the portfolio rate) or the \textit{fixed index system} (the rate credited remains the “new money” rate until the end of period during which this money is treated differently than the entire portfolio).

\textbf{May 2003 SOA/CAS Course 2 Examination, Problem No. 17}

An association had a fund balance of 75 on January 1 and 60 on December 31. At the end of every month during the year, the association deposited 10 from membership fees. There were withdrawals of 5 on February 28, 25 on June 30, 80 on October 15, and 35 on October 31. Calculate the dollar-weighted rate of return for the year.

A. 9.0%  \hspace{1cm} B. 9.55  \hspace{1cm} C. 10.0% \hspace{1cm} D. 10.5% \hspace{1cm} E. 11.0%

Solution.
Remember that the dollar-weighted rate of return calculations effectively assume simple interest. We have:

- Initial balance = 75,
- Final balance = 60,
- Total deposits = 120,
- Total withdrawals = 145,
- Investment income = 60 + 145 – 120 – 75 = 10.

Therefore:
\[
\text{Rate of return} = \frac{10}{75 + \left(\frac{11}{12} + \frac{10}{12} + \cdots + 0\right) \cdot 10 - \frac{10}{12} \cdot \frac{6}{12} - \frac{2.5}{12} \cdot \frac{80}{12} - \frac{2}{12} \cdot \frac{35}{12}} = 11\%.
\]

Answer E.

\textbf{May 2003 SOA/CAS Course 2 Examination, Problem No. 30}

You are given the following table of interest rates:

<table>
<thead>
<tr>
<th>Calendar Year</th>
<th>Investment Year Rates (in %)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Portfolio Rates</td>
</tr>
<tr>
<td></td>
<td>Of Original</td>
</tr>
</tbody>
</table>

\begin{tabular}{|c|c|c|c|c|c|}
\hline
$y$ & $i_1^y$ & $i_2^y$ & $i_3^y$ & $i_4^y$ & $i_5^y$ & $i^{y+5}$ \\
\hline
1992 & 8.25 & 8.25 & 8.4 & 8.5 & 8.5 & 8.35 \\
1993 & 8.5 & 8.7 & 8.75 & 8.9 & 9.0 & 8.6 \\
1997 & 9.5 & 9.5 & 9.6 & 9.7 & 9.7 & \ \\
1998 & 10.0 & 10.0 & 9.9 & 9.8 & \ \\
\hline
\end{tabular}
A person deposits 1000 on January 1, 1997. Let the following be the accumulated value of the 1000 on January 1, 2000:

- $P$: under the investment year method,
- $Q$: under the portfolio yield method,
- $R$: where the balance is withdrawn at the end of every year and is reinvested at the new money rate (investment year method).

Determine the ranking of $P$, $Q$, and $R$.

A. $P > Q > R$
B. $P > R > Q$
C. $Q > P > R$
D. $R > P > Q$
E. $R > Q > P$

Solution.

We have:

$P = 1000 \cdot 1.095 \cdot 1.095 \cdot 1.096 = 1314.13,$

$Q = 1000 \cdot 1.0835 \cdot 1.086 \cdot 1.0885 = 1280.82,$

$R = 1000 \cdot 1.095 \cdot 1.10 \cdot 1.10 = 1324.95.$

Thus, $R > P > Q$.

Answer D.

3. Annuities

A specific cash flow that shows up in many applications is an annuity: a cash flow with payments occurring at equal time intervals. We will start analyzing annuities by looking at those that have level payments. An annuity paying a monetary unit at the end of every year is called an annuity-immediate. An $n$-year annuity-immediate has the present value:

$$a_n = v + v^2 + \ldots + v^n = \frac{1 - v^n}{i}.$$  

Its accumulated value at the end of $n$ years is:

$$s_n = (1 + i)^n - 1.$$  

The above assume each payment to be 1. If the payment is $P$ then the formulas need to be multiplied by $P$.

A basic annuity-due pays a monetary unit at the beginning of each year. An $n$-year annuity-due has the present value:

$$\bar{a}_n = 1 + v + \ldots + v^{n-1} = \frac{1 - v^n}{d}.$$  

Its accumulated value at the end of $n$ years is:
\[ \ddot{s}_n = (1+i)^n + (1+i)^{n-1} + \ldots + (1+i) = \frac{(1+i)^n - 1}{d}. \]

Note that:
\[ \ddot{a}_n = 1 + a_{n-1}, \quad \ddot{s}_n = s_{n+1} - 1, \]
\[ \ddot{a}_n = (1+i)a_{n-1}, \quad \ddot{s}_n = (1+i)s_{n+1}, \]
\[ \frac{1}{a_n} + i, \quad \frac{1}{s_n} = \frac{1}{\ddot{a}_n} + d. \]

The last two identities illustrate that a loan of 1 can be repaid by level payments throughout the \( n \) year period of the loan, or by paying interest every year, and making payments that accumulate to the loan amount at the end of the \( n \) year period of the loan.

A deferred annuity starts payments at a time in the future. If the first payment is made at the time \( n+1 \), and the last one at the time \( n+m \), then its present value is:
\[ n \ddot{a}_{n+m} = n \ddot{a}_{n-1} - a_n. \]

Note also the following \( v^n - v^m = i\left(a_n - a_{n-1}\right) \), and especially:
\[ 1 = v^n + ia_{n-1} \quad (David \: Ricardo \: Identity: \: other \: books \: do \: not \: use \: this \: name) \]

A perpetuity is an annuity that lasts forever.

Perpetuity immediate present value is: \( a_\infty = \lim_{n \to \infty} a_n = \frac{1}{i} \).

Perpetuity due present value is: \( \ddot{a}_\infty = \lim_{n \to \infty} \ddot{a}_n = \frac{1}{d} \).

If the interest rate varies:
\[ a_n = \frac{1}{a(1)} + \frac{1}{a(2)} + \ldots + \frac{1}{a(n)}, \]
\[ s_n = \frac{a(n)}{a(1)} + \frac{a(n)}{a(2)} + \ldots + \frac{a(n)}{a(n)}. \]

If the compounding frequency of the interest exceeds the payment frequency, we need to adjust the annuity formulas. This can be done two ways:
(i) By using equivalent payments made with the same frequency as compounding,
(ii) By finding an equivalent interest rate over the payment period.

If a unit payment is made every \( k \) years at the end of each \( k \) year period, at an annual interest rate of \( i \), then using the two approaches we can either:
(i) Use an annual payment of \( \frac{1}{s_n} \) paid at the end of each year, or
(ii) Use an equivalent interest rate over \( k \) years, denoted by \( j \), given by \( j = (1+i)^k - 1 \).
If the payment frequency exceeds compounding frequency, then the standard approach is to introduce a new concept: an \(m\)-thly annuity. A unit \(m\)-thly annuity immediate pays a total of a monetary unit over a year, with \(\frac{1}{m}\) paid each \(m\)-th of the year, each payment made at the end of the \(m\)-th of the year (if the payment is made at the beginning of the \(m\)-th of the year, this is the \(m\)-thly annuity due), for \(n\) years, and its present value is:

\[ a^{(m)}_n = \frac{1 - v^n}{i^{(m)}} = \frac{i}{i^{(m)}} a^n, \]

while its accumulated value is:

\[ s^{(m)}_n = (1 + i)^n - 1 = \frac{i}{i^{(m)}} s^n. \]

The corresponding formulas for the annuity-due are:

\[ \dot{a}^{(m)}_n = \frac{1 - v^n}{d^{(m)}} = \frac{d}{d^{(m)}} \dot{a}^n, \]

\[ \dot{s}^{(m)}_n = (1 + i)^n - 1 = \frac{d}{d^{(m)}} \dot{s}^n. \]

But the alternative approach is also possible: find an equivalent interest rate \(j\) effective over the payment period, given by: \(j = (1 + i)^{\frac{1}{m}} - 1.\)

A continuous annuity pays a monetary unit per year in a continuous fashion, and it lasts for \(n\) years. Its present value is:

\[ \ddot{a}_n = \frac{1 - v^n}{\delta} = \frac{i}{\delta} a^n, \]

while its accumulated value at the end of \(n\) years is:

\[ \ddot{s}_n = \frac{(1 + i)^n - 1}{\delta} = \frac{i}{\delta} s^n. \]

An alternative approach to the valuation of a continuous annuity is:

\[ \ddot{a}_n = \int_0^n v^t dt = \int_0^n e^{-\delta t} dt. \]

In general, if a payment is made continuously at the annual rate \(P_t\), then the present value of such a stream of payments is:

\[ \int_0^n v^t P_t dt. \]

Varying annuities introduce a new level of complication: varying amounts of payments. If payments increase geometrically, i.e., if each consecutive payment is \((1 + r)\) times the previous payment, then the present value \(V(0)\) of such a stream of payments is:
\[ V(0) = 1 \cdot v + (1 + r) \cdot v^2 + \ldots + (1 + r)^{n-1} \cdot v^n = \frac{1 - \left(\frac{1 + r}{1 + i}\right)^n}{i - r}. \]

A basic increasing annuity immediate makes payments of 1 at time 1, 2 at time 2, etc., with the last payment of \( n \) made at time \( n \). The present value of such an annuity is:

\[(Ia)_{\bar{n}} = \frac{\bar{a}_n - n v^n}{i},\]

while the accumulated value is:

\[(Is)_{\bar{n}} = \frac{s_n - n}{i}.\]

Note also that a basic increasing perpetuity immediate is worth:

\[\lim_{n \to \infty} (Ia)_{\bar{n}} = \lim_{n \to \infty} \frac{\bar{a}_n - n v^n}{i} = \lim_{n \to \infty} \frac{d}{i} = \frac{1}{di}.\]

If we consider an increasing \( n \)-year annuity-due, its present value is:

\[(I\bar{a})_{\bar{n}} = (1 + i) \frac{\bar{a}_n - n v^n}{i} = \frac{\bar{a}_n - n v^n}{iv} = \frac{\bar{a}_n - n v^n}{d},\]

and the present value of an increasing perpetuity-due is:

\[\lim_{n \to \infty} (I\bar{a})_{\bar{n}} = \lim_{n \to \infty} \frac{\bar{a}_n - n v^n}{d} = \lim_{n \to \infty} \frac{d}{d} = \frac{1}{d^2}.\]

A basic decreasing annuity immediate makes payments of \( n \) at time 1, \( n - 1 \) at time 2, etc., with the last payment of 1 made at time \( n \). The present value of such an annuity is

\[(Da)_{\bar{n}} = \frac{n - a_n}{i},\]

while the accumulated value is:

\[(Ds)_{\bar{n}} = \frac{n(1 + i)^n - s_n}{i}.\]

We also have

\[(D\bar{a})_{\bar{n}} = \frac{n - \bar{a}_n}{d},\]

and

\[\left(\bar{D}\bar{a}\right)_{\bar{n}} = \frac{n - \bar{a}_n}{\delta}.\]

Note that:

\[(Da)_{\bar{n}} + (Ia)_{\bar{n}} = (n + 1)a_{\bar{n}}.\]

If the payments of \( \frac{1}{m} \) are made at the end of each \( m \)-th of the first year, \( \frac{2}{m} \) at the end of each \( m \)-th of the second year, etc., for \( n \) years, then the present value of such an annuity is
\[(Ia)^{(m)}_{\overline{n}|} = \frac{\ddot{a}_{n} - nv^{n}}{i^{(m)}}.\]

If the payments at the end of consecutive \(m\)-ths are \(\frac{1}{m^2}\), \(\frac{2}{m^2}\), \(\frac{3}{m^2}\), etc., then the present value of such an annuity is
\[(I^{(m)}a)^{(m)}_{\overline{n}|} = \frac{\ddot{a}_{n}^{(m)} - nv^{n}}{i^{(m)}}.\]

In the limit
\[(\overline{a})(n)_{\overline{n}|} = \frac{\ddot{a}_{n} - nv^{n}}{\delta}.\]

In general, we have:
\[\overline{a}_{\overline{n}|} = \int_{0}^{T} te^{-\delta t} dt = \left[ -t \frac{e^{-\delta t}}{\delta} \right]_{0}^{T} + \left[ \frac{e^{-\delta t}}{\delta} \right]_{0}^{T} = \left( -\frac{T e^{-\delta T}}{\delta} + \frac{\ddot{a}_{\overline{n}|}}{\delta} \right) \]

Therefore,
\[\overline{a}_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - Tv^{T}}{\delta},\]
and
\[\delta(\overline{a})_{\overline{n}|} + Tv^{T} = \ddot{a}_{\overline{n}|}.\]

Let us also note that if a single payment of 1 is made at time \(n\), its present value at time 0 is \(v^{n}\). If a perpetuity of 1 is paid starting at time \(n\), its present value at time 0 is \(\frac{v^{n}}{d}\). If an increasing perpetuity of 1, 2, 3, … is paid starting at time \(n\), its present value at time 0 is \(\frac{v^{n}}{d^{2}}\). These three formulas may be useful for analyzing various cash flows.

**May 2003 SOA/CAS Course 2 Examination, Problem No. 8**
Kathryn deposits 100 into an account at the beginning of each 4-year period for 40 years. The account credits interest at an annual effective interest rate of \(i\). The accumulated amount in the account at the end of 40 years is \(X\), which is 5 times the accumulated amount in the account at the end of 20 years. Calculate \(X\).

A. 4695  B. 5070  C. 5445  D. 5820  E. 6195

**Solution.**
The effective interest rate over a four-year period is:
\[j = (1 + i)^{4} - 1.\]
Using that rate, we have:
\[ X = 100 \ddot{s}_{10|i} = 100 \frac{(1 + j)^{10} - 1}{j} = 5 \cdot 100 \ddot{s}_{j} = 500 \frac{(1 + j)^{5} - 1}{j} \cdot \frac{1}{1 + j}. \]

Therefore (note that the subscript \( j \) refers to the interest functions involving the interest rate \( j \)),
\[
5 = \frac{(1 + j)^{10} - 1}{(1 + j)^{5} - 1} = (1 + j)^{5} + 1.
\]

Thus
\[
(1 + j)^{5} = 4,
\]
and \( j = 31.9508\% \). Also
\[
d_{j} = \frac{j}{1 + j} = 24.2142\%.
\]

Therefore:
\[
X = 100 \ddot{s}_{10|i} = 100 \frac{(1 + j)^{10} - 1}{d_{j}} = 100 \frac{4^{2} - 1}{0.242142} = 6194.72.
\]

Answer E.

May 2003 SOA/CAS Course 2 Examination, Problem No. 22
A perpetuity costs 77.1 and makes annual payments at the end of the year. This perpetuity pays 1 at the end of year 2, 2 at the end of year 3, \ldots, \( n \) at the end of year \((n+1)\). After year \((n+1)\), the payments remain constant at \( n \). The annual effective interest rate is 10.5%. Calculate \( n \).

A. 17  B. 18  C. 19  D. 20  E. 21

Solution.
The cost of this perpetuity =
\[
v \cdot (l_{a})_{n} + \frac{n + v^{n+1}}{i} = v \left( \frac{\ddot{a}_{n} - nv^{n}}{i} \right) + \frac{n \cdot v^{n+1}}{i} = \frac{a_{n}}{i} - \frac{nv^{n+1}}{i} + \frac{nv^{n+1}}{i} = \frac{a_{n}}{i}.
\]

Since \( i = 10.5\% \), we have
\[
\frac{a_{n}}{i} = \frac{a_{n}}{0.105} = 77.10,
\]
and \( a_{n} = 8.0955 \) at 10.5\%. Hence \( n = 19 \).

Answer C.

May 2003 SOA/CAS Course 2 Examination, Problem No. 26
1000 is deposited into Fund X, which earns an annual effective rate of 6\%. At the end of each year, the interest earned plus an additional 100 is withdrawn from the fund. At the end of the tenth year, the fund is depleted. The annual withdrawals of interest and
principal are deposited into Fund Y, which earns an annual effective rate of 9%.
Determine the accumulated value of Fund Y at the end of year 10.

A. 1519     B. 1819     C. 2085     D. 2273     E. 2431

Solution.
The interest earned in Fund X during the first year is 60. The second year, it is reduced by 6, the interest on the 100 withdrawn. Next year, it is again reduced by 6, and this continues until the end of the tenth year. Thus the interest portion of withdrawals, when treated as deposits into Fund Y amount to a decreasing ten-year annuity, which is six times the unit ten-year decreasing annuity. Therefore the total accumulated in Fund Y is:

$$6(Ds)_{\overline{10}|0.09} + 100s_{\overline{10}|0.09} = 6 \left( \frac{10 \cdot 1.09^{10} - s_{\overline{10}|0.09}}{0.09} \right) + 100 \cdot 15.19293 =$$

$$= 565.38 + 1519.29 = 2084.67.$$

Answer C.

May 2003 SOA/CAS Course 2 Examination, Problem No. 45
A perpetuity-immediate pays 100 per year. Immediately after the fifth payment, the perpetuity is exchanged for a 25-year annuity-immediate that will pay X at the end of the first year. Each subsequent annual payment will be 8% greater than the preceding payment. Immediately after the 10-th payment of the 25-year annuity, the annuity will be exchanged for a perpetuity-immediate paying Y per year. The annual effective rate of interest is 8%. Calculate Y.

A. 110     B. 120     C. 130     D. 140     E. 150

Solution.
The value of the perpetuity is $\frac{100}{0.08} = 1250$. The value of the 25-year annuity-immediate it is exchanged for is:

$$1250 = X \cdot 1.08^{-1} + X \cdot 1.08 \cdot 1.08^{-2} + \cdots + X \cdot 1.08^{24} \cdot 1.08^{-25} = \frac{25X}{1.08}$$

This implies that $X = 54$. When the second exchange happens, the equation of value is:

$$\frac{Y}{0.08} = 54 \cdot 1.08^{10} \cdot 1.08^{-1} + 54 \cdot 1.08^{11} \cdot 1.08^{-2} + \cdots + 54 \cdot 1.08^{24} \cdot 1.08^{-15} =$$

$$= 54 \cdot 1.08^9 \cdot 15.$$

This solves for $Y = 129.5$.
Answer C.

4. Loan Amortization
Let us assume that a hypothetical borrower repays a lender by a series of payments at regular intervals, at a fixed interest rate, with the following notation:
$L$ = amount of the loan,
\( n \) = number of payments,
\( P \) = amount of the level payment, paid at the end of each period,
\( i \) = effective interest rate per payment period (typically a year, although for mortgages in the U.S., payments are made monthly, while for bonds issued in the U.S., semi-annual payments are a norm – in all other countries around the world annual payments on business loans and bonds are a norm).

With the above notation, we have:

\[
L = P a_{\frac{n}{i}},
\]

and

\[
P = \frac{L}{a_{\frac{n}{i}}}.
\]

The balance of the loan immediately after the \( k \)-th payment is:

\[
B_k = P a_{\frac{n-k}{i}} \text{ (prospectively)} = L (1 + i)^k - P a_{\frac{n-k}{i}} \text{ (retrospectively)}.
\]

Also note that

\[
B_{k+1} = (1 + i)^t B_k
\]

for fractional \( t \).

Each payment can be split into a portion paying the interest \( I_k \) and the portion paying the principal \( P_k \), so that \( P = P_k + I_k \), and

\[
I_k = i B_{k-1} = P i \frac{1 - v^{n-(k-1)}}{i} = P (1 - v^{n-k}) = P - P_k,
\]

\[
P_k = P - I_k = P v^{n-k}.
\]

One very important, crucial, observation is that the ratio of the principal portions of two consecutive payments is always \( 1 + i \) or \( v \) (depending on which way we divide them, but since we assume working with positive interest rates, you can always see which of the two you are getting as an answer, even if you set up your division in the wrong order).

If loan payments are not level, amortization has to be done step by step. Suppose that payments are: \( P_{(1)}, P_{(2)}, \ldots, P_{(n)} \) at times \( 1, 2, \ldots, n \). Then we have

\[
L = P_{(1)} v + P_{(2)} v^2 + \ldots + P_{(n)} v^n,
\]

\[
B_k = P_{(k+1)} v + P_{(k+2)} v^2 + \ldots + P_{(n)} v^{n-k} =
\]

\[
= L (1 + i)^k - P_{(1)} (1 + i)^{k-1} - \ldots - P_{(k)} = B_{k-1} (1 + i) - P_{(k)}.
\]

where the last version of the balance formula, the recursive one, is especially useful for such nonstandard loans. Furthermore:

\[
I_k = i B_{k-1}
\]

and

\[
P_k = P_{(k)} - I_k.
\]

The above amortization procedure is standard for mortgage bonds in the U.S., especially individual mortgages for residential homes (although payments are typically made
monthly for these). For bonds issued in the U.S., it is standard to only make interest payments between issue date and maturity date, with full repayment of principal at maturity date. However, in case of some bonds, especially issued by local governments (cities, counties and states) it is common to accumulate money for the principal repayment in a separate fund, called a *sinking fund*, by making a payment, in addition to the regular interest payment, every period. Let us use the following notation:

\[ L = \text{amount of the loan}, \]
\[ n = \text{number of payment periods}, \]
\[ i = \text{effective interest rate per payment period paid by the borrower to the lender}, \]
\[ j = \text{effective interest rate earned by the borrower on the sinking fund}, \]
\[ D = \text{periodic sinking fund deposit (assumed level)}, \]
\[ P = \text{periodic outlay by the borrower} = \text{interest payment to lender} + \text{sinking fund deposit}. \]

Then we have the following:

\[ L = D s_n^j, \quad D = \frac{L}{s_n^j}, \]
\[ P = Li + D = Li + \frac{L}{s_n^j} = \frac{L}{\frac{a_n^j}{1 + (i - j) a_n^j}}. \]

Often denoted by \(a_{-n}^j\)

Net loan balance at time \(k\) (after consideration for the accumulation in the sinking fund):

\[ L - D s_k^j = \text{Loan Balance} – \text{Sinking Fund Balance}. \]

Principal and interest portions of the \(k\)-th payment:

- Net Principal Repaid = \(D s_k^j - D s_{k-1}^j = D(1 + j)^{k-1}\),
- Net Interest Paid = \(Li - jDs_{k-1}^j\).

Special case when \(i = j\)

\[ P = Li + \frac{L}{s_n^j} = L \left( i + \frac{1}{s_n^j} \right) = \frac{L}{a_n^j}, \]

i.e., the payment is exactly the same as in the case of level amortization discussed previously. Also, net interest is the same as interest payment in that previous method, and principal repaid (which is the increase in the sinking fund balance in this method, and principal repaid in the previous one) are the same as before.

Now let us compare the sinking fund method with \(i > j\) and level amortization discussed previously at the loan interest rate \(j\). In this case:

- Net amount of the loan after the \(k\)-th period is the same for both methods:

\[ L - D s_k^j = L - \frac{L}{s_n^j} s_k^j = L \left( s_n^j - s_k^j \right) = L \left( \frac{a_n^j - v^{n-k} a_n^j}{a_n^j} \right) = \frac{L}{a_n^j} a_{-n}^j. \]
Net interest in the $k$-th period is equal to $L(i - j)$ plus the amount of interest in the $k$-th payment in the amortization method:

$$Li - jDs_{k-1}|j} = L(i - j) + Lj - jDs_{k-1}|j} = L(i - j) + j\left(\frac{L - Ds_{k-1}|j}}{p_{n-k+1}|j}\right),$$

where $P$ is the amortization method payment.

The increase in the balance of the sinking fund in the $k$-th period is the same as the principal repaid in the $k$-th period in the amortization method:

$$Ds_{k-1}|j} - Ds_{k-1}|j} = D(1 + j)^{k-1} = \frac{L}{s_{n|j}}(1 + j)^{k-1} \frac{v^n}{a_{n|j}} = \frac{L}{v^n} v^{n-k+1} = P_{1}^{n-k+1}.$$ Note that in all discussions above, we always had the same phenomenon that the ratio of two consecutive principal repayments, in any methodology, is always either the discount factor, or one plus the interest rate, at the interest rate applicable to the principal.

**May 2003 SOA/CAS Course 2 Examination, Problem No. 15**

John borrows 1000 for 10 years at an annual effective interest rate of 10%. He can repay this loan using the amortization method with payments of $P$ at the end of each year. Instead, John repays the 1000 using a sinking fund that pays an annual effective rate of 14%. The deposits to the sinking fund are equal to $P$ minus the interest on the loan and are made at the end of each year for 10 years. Determine the balance in the sinking fund immediately after repayment of the loan.

A. 213  B. 218  C. 223  D. 230  E. 237

**Solution.**

We have

$$a_{\overline{10}|0}\% \approx 6.14457.$$ Therefore, the payment using the amortization method is

$$\frac{1000}{a_{\overline{10}|0}\%} = \frac{1000}{6.14457} = 162.745.$$ The interest on the loan amount of 1000 is $1000 \cdot 10\% = 100$. Hence, the deposits into the sinking fund equal $162.745 - 100 = 62.745$. At 14% we have $s_{\overline{10}|14} = 19.3373$. The accumulated value of the sinking fund is $62.745 \cdot 19.3373 = 1213.319$. The balance in the sinking fund after repayment of the loan will be: $1213.319 - 1000 = 213.319$. Answer A.

**May 2003 SOA/CAS Course 2 Examination, Problem No. 33**

At an annual effective interest rate of $i$, $i > 0$, both of the following annuities have a present value of $X$:

(i) a 20-year annuity-immediate with annual payments of 55,
(ii) a 30-year annuity-immediate with annual payments that pay 30 per year for the first
10 years, 60 per year for the second 10 years, and 90 per year for the final 10 years. Calculate X.

A. 575  B. 585  C. 595  D. 605  E. 615

Solution.
We have
\[ 55a_{20|i} = 55 \cdot \frac{1 - v^{20}}{i} = 55 \cdot \left(1 - v^{10}\right)\left(1 + v^{10}\right) = 55 \cdot a_{10|i} \cdot (1 + v^{10}) = 30 \cdot a_{10|i} + 60 \cdot v^{10} \cdot a_{10|i} + 90 \cdot v^{20} \cdot a_{10|i} = a_{10|i} \cdot (30 + 60v^{10} + 90v^{20}). \]
Therefore
\[ 55 + 55v^{10} = 30 + 60v^{10} + 90v^{20}, \]
or
\[ 90v^{20} + 5v^{10} - 25 = 0, \]
and this quadratic equation in \( z = v^{10} \) has two solutions:
\[ z = v^{10} = \frac{-5 \pm \sqrt{25 + 9000}}{180}, \]
but only one of them positive \( v^{10} = \frac{90}{180} = 0.5 \). We could find the interest rate from this
\[ i = 7.18\%, \text{ and hence } X = 55 \cdot a_{50|i,7.18\%} = 574.60. \]
Answer A.

May 2003 SOA/CAS Course 2 Examination, Problem No. 39
A 30-year loan of 1000 is repaid with payments at the end of each year. Each of the first ten payments equals the amount of interest due. Each of the next ten payments equals 150% of the amount of interest due. Each of the last ten payments is \( X \). The lender charges interest at an annual effective rate of 10%. Calculate \( X \).

A. 32  B. 57  C. 70  D. 97  E. 117

Solution.
Since the first ten payments equal the annual interest due, the amount of principal outstanding at the end of 10 years is the amount of loan: 1000. For the next 10 years, each payment equals 150% of interest due. The lender charges 10%, therefore 5% of the principal outstanding will be used to reduce the principal. At the end of 20 years, the amount outstanding is
\[ 1000(1 - 0.05)^{10} = 598.74. \]
Each of the last ten payments is:
\[ \frac{598.74}{a_{10|i,5\%}} = 97.4417. \]
Answer D.

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5. Bonds and other securities

Bonds are interest-bearing securities, promising to pay a specific rate of return. Bonds are investment assets of the lender, and loans to the borrower. Bonds usually pay interest in a form of a coupon, a periodic interest payment, but some bonds accrue interest without paying it out, and then make a payment at the end of the term of the bond (maturity) – such bonds without coupons are called zero-coupon bonds, or zeros, or pure accumulation bonds. A bond whose payments are backed by a property (such as a building) is called a mortgage bond, a bond without such backing is called a debenture. Bonds issued by the United States Treasury are called Treasury Bills if their maturity is a year or less, Treasury Notes if the maturity is more than a year but less than ten years, and Treasury Bonds if their term is ten years or more. A bond is callable if it can be paid off before maturity, otherwise it is called noncallable.

A preferred stock is a security that pays a prescribed level of a dividend in perpetuity, and represents right to ownership of the underlying company, but usually without voting rights, and with only limited ownership. Real ownership is given by common stock, whose dividend varies.

A bond or preferred stock than can be exchanged for a specific number of shares of a common stock is called convertible.

We will now consider bonds with fixed coupons, by far the most common in the United States, uncommon in other countries. We will use the following notation:

\( F \) = Par value of the bond (amount given on the bond certificate, may not equal to the actual amount borrowed or repaid, but usually is equal to the principal repaid at maturity).

\( r \) = Stated coupon rate of the bond (so that the coupon amount is \( Fr \)).

\( C \) = Redemption value of the bond (commonly equal to \( F \)).

\( n \) = Number of coupon periods to maturity.

\( P \) = Market price of the bond.

\( i \) = Bond yield, calculated as the internal rate of return on bond’s cash flows, given its market price.

\( g = \frac{Fr}{C} \) = modified coupon rate.

\( K = Cv^n \) = Present value of the redemption value calculated at the yield \( i \).

If \( P > C \) the bond is said to sell at a premium, and \( P - C \) is its premium. If \( P < C \), then the bond is said to sell at a discount, and \( C - P \) is the amount of its discount.

We also write

\( k = \frac{P - C}{C} \).

If the bond is selling at the price \( P = C \) we say that it sells at par.

The formulae for the price of a bond in relation to its yield are:

\[ P = Fr \cdot \frac{i}{n} + K \]“Frank” Formula

\[ P = K + \frac{g}{i} (C - K) \]Makeham’s Formula
\[ P = C \left(1 + (g - i) a_n\right) \]  

*Premium-Discount Formula*, called so because it implies that  
\[ k = (g - i) a_n. \]

If we define  
\[ G = \frac{Fr}{i}, \]
the *base amount* of the bond, then we also have this  
\[ P = G + (C - G) v^n \]  

*Base Amount Formula.*

If a bond is bought at a discount, it will still be redeemed at par. Thus the investor who bought this bond will enjoy price appreciation between the time of purchase and redemption. But the bond is selling at a discount because its coupon is below the current market rate. Thus the investor is getting a coupon, which is too low in relation to the current market level of interest rates, but he/she is properly compensated for this by having the bond increase in value until it reaches full redemption value at maturity. This process of marking up the price of a bond bought at a discount is called the *amortization of discount.* If a bond is bought at a premium, it will still pay only the regular redemption value at maturity, thus it will gradually decline in value between the time of purchase and maturity. This will, however, be accompanied by payments of coupons in excess of current market level of interest rates, which was the reason why the bond sold at a premium in the first place. Overall, any kind of a bond, one sold at a discount, one sold at par, or one sold at a premium, will offer the same yield as the current market level of appropriate yields, and the whole process balances itself. We should make sure that we understand the process of premium/discount amortization. The *book value* of a bond at time \( t \) is the value at time \( t \) of the remaining future cash flows at the yield rate. Let us denote it by \( BV_t \). The terminology is related to the fact that this used to be the value of a bond shown in accounting statements of financial institutions, and still is the value shown in the statutory financial statement of U.S. insurance companies (also in the Generally Accepted Accounting Principles, i.e., GAAP, statement, if the bond is designated as held-to-maturity). In particular, \( BV_k \) is the book value immediately after the \( k \)-th coupon is paid. We have:  
\[ BV_k = Fr a_n - k + C v^{n-k} = v \cdot Fr + v \cdot BV_{k+1}. \]

Note that the second part of the above equation gives us the recursive formula relating the book values of a bond between two consecutive coupon payments. We also have  
\[ BV_k (1 + i) = Fr + BV_{k+1}. \]

The valuation of bonds between coupon dates gets to be a bit more complicated. If a bond is bought/sold between coupon payment dates, a problem of allocation of the upcoming coupon between the current owner (selling the bond) and the new owner. Note that at the time of the next coupon payment, the entire coupon will be collected by the new owner. A certain part of it is assigned to the current owner, and collected by him/her at sale. In order to explain this process, we present some new terminology. The compensation for the portion of the coupon attributable to the portion of the coupon period before the date of the sale (denoted here by \( t + k \), with \( 0 < k < 1 \), and \( t \) being the time of one coupon payment, and \( t + 1 \) being the time of the next coupon payment, sorry about changing notation for this case, and using \( t \) instead of \( k \), as \( k \) will now be a fraction of the year), is
called the *accrued coupon*, and is denoted by $Fr_k$. We have $Fr_0 = 0$, and $Fr_1 = Fr$. The *flat price* is the amount of money which actually changes hands at the time of the sale and is denoted by $B_{t+k}^f$. The *market price* of a bond is the price excluding accrued coupon and is denoted by $B_{t+k}^m$. Then

$$B_{t+k}^f = B_{t+k}^m + Fr_k.$$  

There are three methods of calculation of the pieces of the above formula:

- **Theoretical Method**
  $$B_{t+k}^f = B_t (1 + i)^k,$$
  $$Fr_k = Fr \cdot \frac{(1 + i)^k - 1}{i},$$
  $$B_{t+k}^m = B_t (1 + i)^k - Fr \cdot \frac{(1 + i)^k - 1}{i}.$$  

- **Practical Method**
  $$B_{t+k}^f = B_t (1 + ki),$$
  $$Fr_k = kFr,$$
  $$B_{t+k}^m = B_t (1 + ki) - kFr.$$  

- **Semi-Theoretical Method**
  $$B_{t+k}^f = B_t (1 + i)^k,$$
  $$Fr_k = kFr,$$
  $$B_{t+k}^m = B_t (1 + i)^k - kFr.$$  

We also see that the change in book value between two consecutive coupon payments is:

$$\Delta BV_k = BV_{k+1} - BV_k = iBV_k - Fr.$$  

This change, if positive, is the amount of the “write-up” in the bond value during the $k$-th period, if the bond was sold at a discount, and if negative, its opposite is the amount of the “write-down” in the $k$-th period, if the bond was sold at a premium. Note also that $BV_0 = P$ and $BV_n = C$. We can either use the general formula for finding the book values in between, or use the two boundary values and the recursive relationship to arrive at the same intermediate book value.

Let us look at the process of premium/discount amortization of a bond. The initial price of a bond is:

$$B_0 = C \left(1 + (g - i)a_{\bar{n}}\right).$$  

When the coupon is paid at time 1, its value is $Fr$, while the amount of interest is

$$iC \left(1 + (g - i)a_{\bar{n}}\right).$$  

The new price is

$$B_1 = C \left(1 + (g - i)a_{\bar{n-1}}\right) = C \left(1 + (g - i)a_{\bar{n}}\right) - C(g - i)v^n = B_0 - C(g - i)v^n.$$  

The change in price is

$$C(g - i)v^n.$$
Let us compare it with the difference between the coupon and the level of interest corresponding to $i$:

$$Fr - iC \left(1 + (g - i) a_{\frac{1}{i}}\right) = Fr - Ci - C (g - i) \cdot i \cdot \frac{1 - v^n}{i} =$$

$$= Fr - Ci - Cg (1 - v^n) + Ci (1 - v^n) = Fr - Ci + Cg + Cg v^n + Ci - Ci v^n = C (g - i) v^n.$$

Thus the coupon can be viewed as being split into a portion that pays the rate of interest $i$, i.e.,

$$iC \left(1 + (g - i) a_{\frac{1}{i}}\right) = iC + (g - i) C (1 - v^n) =$$

$$= gC - C (g - i) v^n = Fr - C (g - i) v^n,$$

and a portion that amortizes the premium or discount, equal to $C (g - i) v^n$. The above reasoning can be applied to the bond at any point in time immediately following a coupon payment. As always, the principal amortization portions of consecutive coupons follow the pattern of two consecutive values having the ratio of $v$. To be more specific, the principal adjustments and interest in the coupons follow this pattern:

<table>
<thead>
<tr>
<th>Principal</th>
<th>Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C (g - i) v^n$</td>
<td>$Fr - C (g - i) v^n$</td>
</tr>
<tr>
<td>$C (g - i) v^{n-1}$</td>
<td>$Fr - C (g - i) v^{n-1}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$C (g - i) v$</td>
<td>$Fr - C (g - i) v$</td>
</tr>
</tbody>
</table>

Note also that the sum of all principal adjustments is:

$$\sum_{k=1}^{n} C (g - i) v^k = C (g - i) a_{\frac{1}{i}}.$$

In what we stated above, we have always assume that the principal repayments in a bond occur at exactly the scheduled times. But in reality, while that is the case for almost all bonds issued by the United States Federal Government, other bonds issued in the United States, be it corporate bonds (i.e., bonds issued by corporations), or municipal bonds (bonds issued by states, cities, and other local levels of government), do not have this property. Mortgages (i.e., loans for which the borrower pledges real property as a collateral) for homes purchased by individuals and families can always be paid off early. So can corporate bonds and municipal bonds. While home mortgages can be paid off at any time, corporate and municipal bonds have prescribed call schedules, i.e., times and prices at which they can be paid off. A possible exam problem is a question of what the yield to the investor is if a bond is called at a specific time. The price at which a corporate or a municipal bond can be paid off early in full, or in prescribed percentage of the issue, is termed the call price, and it does not have to equal the market price of the bond, or its par value, or its redemption value. Rather, it is prescribed in the bond covenants, in advance. Having a bond or a mortgage called, or paid off early, is almost never a happy news for the investor, as the payoff usually happens when interest rates are relatively low and the investors are stuck with low reinvestment yield.
One more risk faced by the investors is that the principal may not arrive on time, or early, but rather late or never at all. If the borrower misses a payment of interest and/or principal, such borrower is said to be in default, and the matter is referred to a bankruptcy court. The investor generally recovers some portion of principal from the assets of the borrower, but such recovery becomes very uncertain, and the recovery rate is almost never near 100% (historical average recovery rates are in the range of 40% or less). The risk of default by the borrower is termed the credit risk.

May 2003 SOA/CAS Course 2 Examination, Problem No. 42

A 10,000 par value 10-year bond with 8% annual coupons is bought at a premium to yield an annual effective rate of 6%. Calculate the interest portion of the 7-th coupon.

A. 632  B. 642  C. 651  D. 660  E. 667

Solution.
The interest rate is $i = 6\%$. Using it we have:

$$BV_6 = \frac{10000}{1.06^4} + 800a_{4.06}^i = \frac{10000}{1.06^4} + 800 \left( \frac{1}{1.06} \right)^4 = 7920.94 + 2772.08 = 10693.$$ 

Thus the interest portion of the seventh coupon is:

$$I_7 = i \cdot BV_6 = 0.06 \cdot 10693 = 641.58.$$ 

The principal portion will be $800 - 641.58 = 158.42$. Next book value will be $10693 - 158.42 = 10534.58$.

Answer B.

6. Duration

The method of presentation of ideas of duration and convexity in this manual will differ slightly from what you see in most textbooks, but it results in the same practical approaches.

Duration is a measure of sensitivity of a financial asset to changes in interest rates. It is based on the assumption of using only one interest rate, which is commonly interpreted as a “flat yield curve” assumption. Since changes in one interest rate amount to a parallel shift in such a flat yield curve, use of duration is also commonly said to assume a “parallel shift” in the yield curve. Let us write $P(i)$ for the price of a financial asset as a function of the interest rate $i$, and $P(\delta)$ for the same price, when it is considered a function of the force of interest, with the natural relationship of the interest rate and the force of interest: $1 + i = e^\delta$. The duration of this security is defined as:

$$D(P(i)) = -\frac{d}{di} \left( \frac{1}{P(i)} \right) = -\frac{d}{di} \ln(P(i)).$$

Mathematically, duration is the opposite of the logarithmic derivative of the price of a security with respect to the interest rate. Because of the standard approximation of the derivative with a difference quotient we see that:
This means that duration gives us the ratio of the percentage loss in the value of the security per unit of interest rates, a very commonly used approximation. Note also that since the loss in the value of the security

\[
P(i - \Delta i) - P(i)
\]

is expressed as percentage, and \( \Delta i \) is in percent per year (if the interest rate used is annual, a common standard), the unit for duration is a year (or the time unit over which the interest rate is given).

The opposite of the derivative of the price with respect to the interest rate

\[
-\frac{dP(i)}{di}
\]

is usually termed the dollar duration of the security. We would like to use an alternative name: monetary duration, which we believe to be better, as it is universal, without reference to a specific national currency.

One could ask why we chose the derivative with respect to the interest rate, and not with respect to the force of interest for the measure of sensitivity. The two notions are, of course, connected through the relation

\[
1 + i = e^\delta.
\]

Note also that:

\[
\frac{di}{d\delta} = e^\delta = 1 + i,
\]

and

\[
\frac{d\delta}{di} = \frac{1}{1 + i} = v = e^{-\delta}.
\]

The measure of sensitivity with respect to the force of interest:

\[
D_m(P(\delta)) = -\frac{dP(\delta)}{d\delta} \frac{1}{P(\delta)} = -\frac{d}{d\delta} \ln(P(i)) = -(1 + i) \frac{dP(i)}{di} \frac{1}{P(i)} = (1 + i) D(P(i))
\]

will be termed, for the reasons that we will explain momentarily, the Macaulay duration. Observe that

\[
D(P(i)) = \frac{1}{1 + i} D_m(P(\delta)).
\]

Assume now that we know exactly the cash flow \( A_t \) produced by a security under consideration at time \( t \). At first, we will assume that the cash flows are discrete (as opposed to continuous – in fact, securities existing in reality, are effectively all discrete). Assume that there is only one interest rate regardless of maturity (i.e., the yield curve is flat) and that \( A_t \) does not depend on \( i \) (we will say that a security has deterministic cash
flows when its cash flows do not depend on interest rate). Then the present value of the security, its price, is

\[ P(i) = \sum_{t>0} \frac{A_t}{(1+i)^t}. \]

By simply taking the derivative, we see that the monetary (dollar) duration is given by:

\[ -\frac{dP}{di} = \sum_{t>0} \frac{tA_t}{(1+i)^t+1} = \frac{1}{(1+i)} \sum_{t>0} t \cdot \text{PV}(A_t), \]

where \( \text{PV}(A_t) \) is the present value of the cash flow \( A_t \). The duration of this security is therefore:

\[ D(P(i)) = \frac{1}{P(i)} \sum_{t>0} \frac{tA_t}{(1+i)^t+1} = \frac{1}{(1+i)} \sum_{t>0} t \cdot \frac{\text{PV}(A_t)}{P(i)}. \]

If we introduce the weights

\[ w_t = \frac{\text{PV}(A_t)}{P(i)}, \]

then duration turns out to be a weighted-average time to maturity, modified by the factor \( \frac{1}{1+i} \). For that reason, the concept of duration as introduced here in the most general sense, is commonly called modified duration for securities with deterministic cash flows.

For securities with cash flows dependent on interest rates, which causes the cash flows to be random in nature if interest rates are random (as a stochastic process is the most reasonable model of rates evolution), duration is most often termed effective duration. However, in all those cases, duration is the opposite of the logarithmic derivative with respect to the interest rate (although in the estimation of such a derivative in the case of random interest rate, we effectively ignore this randomness and treat the interest rate as a regular variable).

The weighted average time to maturity concept is actually the original idea of duration. Macaulay (in 1938) defined, for a security with deterministic cash flows:

\[ D_M(P(i)) = \frac{1}{P(i)} \sum_{t>0} \frac{tA_t}{(1+i)^t+1} = \sum_{t>0} t \cdot \frac{\text{PV}(A_t)}{P(i)}. \]

We can also see easily that for a security with deterministic cash flows

\[ P(\delta) = \sum_{t>0} e^{-\delta t} \cdot A_t \]

and

\[ D_M(P(\delta)) = -\frac{1}{P(\delta)} \frac{dP(\delta)}{d\delta} = -\frac{d}{d\delta} (\ln P(\delta)) = \sum_{t>0} t \cdot \frac{A_t e^{-\delta t}}{P(\delta)} = \sum_{t>0} t \cdot \frac{\text{PV}(A_t)}{P(\delta)}. \]

If we recall the definition of the weights \( w_t = \frac{\text{PV}(A_t)}{P(i)} \), and introduce a discrete probability distribution with \( \text{Pr}(T=t) = w_t \), then we can also see that the Macaulay duration is the expected value of this probability distribution. We can think of the time \( T \)
to payment as a random variable described by that distribution, and the expected time to payment \( E(T) \) is therefore the Macaulay duration.

Let \( \text{Dur} (P) \) be either the duration or Macaulay duration of any security, whose prices is treated a function of interest rate (or force of mortality) \( P = P(i) = P(\delta) \).

Suppose that and \( P, P_1, \) and \( P_2 \) are prices of securities expressed as functions of interest rates such that:

\[
P(i) = P(\delta) = P_1(i) \pm P_2(i) = P_1(\delta) \pm P_2(\delta).
\]

Then it follows directly from the definition of duration, or Macaulay duration that:

\[
\text{Dur} (P) = \frac{P_1}{P} \text{Dur} (P_1) \pm \frac{P_2}{P} \text{Dur} (P_2) = \frac{P_1}{P_1 \pm P_2} \text{Dur} (P_1) \pm \frac{P_2}{P_1 \pm P_2} \text{Dur} (P_2).
\]

We can quickly see also, that for a single payment at a time \( t \) in the future, its Macaulay duration is \( t \), and its duration is \( \frac{t}{1+i} \). Since duration of a portfolio can be calculated as a weighted average of the duration of individual payments, for securities whose cash flows do not depend on interest rates calculation of duration is actually quite easy.

If the security under consideration is an annuity-immediate paid with certainty for \( n \) years, paid \( m \) times a year, with each payment equal to \( \frac{1}{m} \), then the Macaulay duration of this security is

\[
\frac{1}{d^{(m)}} = \left( \frac{n}{(1+i)^m - 1} \right).
\]

In fact, the cash flows of this security are all \( \frac{1}{m} \), and they come at times \( \frac{1}{m}, \frac{2}{m}, \ldots, \frac{nm}{m} = n \), its price is \( a^{(m)}_{\frac{1}{m}} \), and thus its duration is

\[
\frac{1}{a^{(m)}_{\frac{1}{m}}} \sum_{k=1}^{nm} \frac{k}{m} \cdot \frac{1/v^n}{m} = \frac{1}{d^{(m)}} \frac{d^{(m)} - n v^n}{d^{(m)} - n v^n} = \frac{1 - v^n}{1 - v^n} = \frac{n}{(1+i)^m - 1}.
\]

Let us also observe that if the security is a risk-free bond with principal value of a monetary unit, maturing \( n \) years from now, paying an equal coupon of \( \frac{i^{(m)}}{m} \), \( m \) times a year at the end of each \( m \)-th of a year, with \( i^{(m)} \) being the nominal annual interest rate compounded \( m \) times a year at the time of bond issue, then its price is...
\[ P(\delta) = \hat{i}^{(m)} a_{\bar{n}}^{(m)} + \nu^n = 1, \]

and its Macaulay duration (calculated here as a weighted-average time to maturity) is:

\[
D_{\bar{n}}(P(\delta)) = \frac{\sum_{k=1}^{n^m} \left( k \cdot \frac{i^{(m)}}{m^{(m)}} \cdot v^m \right) + n \cdot \nu^n}{1} = \frac{i^{(m)} \sum_{k=1}^{n^m} k \cdot v^m}{1} + n \cdot \nu^n = \]

\[
= i^{(m)}(Ia)_{\bar{n}}^{(m)} + \nu^n = i^{(m)}(Ia)_{\bar{n}}^{(m)} - n\nu^n = \] 

\[\hat{a}_{\bar{n}}^{(m)} + n\nu^n = \hat{a}_{\bar{n}}^{(m)}.\]

One important comment is that investment professionals, who do not deal with liabilities, often think about duration somewhat differently, as illustrated in the figure below:

In practice, securities have very complex relationship of cash flows to interest rates, and one can’t write a direct functional relationship between the cash flows and interest rate. For that reason, for securities with embedded options (such as an option to repay a loan early, or prepay a home mortgage, or call a bond) duration is usually not calculated, but estimated. Consider the Taylor series expansion of the price as a function of interest rate:

\[
P(i + \Delta i) = P(i) + \frac{dP}{di} \Delta i + \frac{1}{2} \frac{d^2 P}{di^2} (\Delta i)^2 + ... \]

Ignoring nonlinear terms

\[
-\frac{dP}{di} \approx \frac{P(i) - P(i + \Delta i)}{(\Delta i)P(i)} \]

and

\[
-\frac{dP}{di} \approx \frac{P(i - \Delta i) - P(i)}{(\Delta i)P(i)}. \]

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By averaging the above two formulas we obtain a commonly used approximation of duration:

\[ D \approx \frac{P(i - \Delta i) - P(i + \Delta i)}{2P(i)(\Delta i)}. \]

Because this approximation can account for any interest rate and default options embedded in the security, this concept of duration is often called an option-adjusted duration, or effective duration.

**Exercise 6.1**
You are an investment actuary for Honorable Life Insurance Company. Your company has just purchased a callable bond, which at issue had a maturity of 15 years, and 5 years of call protection. You did not buy this bond at issue, but in the secondary market, a while after its issue. You paid a price of 101.42 (per 100 of principal). You are given the following information:

- If interest rates decrease 10 basis points (1 basis point = 0.01%), then the price = 101.58.
- If interest rates increase 10 basis points, then the price = 101.20.

You are also given that the force of interest equals 4% now. Calculate the approximate value of Macaulay duration and effective duration of the bond.

**Solution.**
First, we need to find the interest rate

\[ i = e^\delta - 1 = e^{0.04} - 1 \approx 4.081077\%. \]

The approximation of effective duration is:

\[ D \approx \frac{P(i - \Delta i) - P(i + \Delta i)}{2P(i)(\Delta i)} = \frac{101.58 - 101.20}{2 \cdot 101.42 \cdot 0.001} = \frac{0.38}{0.20284} \approx 1.8734. \]

Macaulay duration = (Effective duration) times (1 + i)

\[ D_M = D(1 + i) = 1.8734 \cdot 1.04081077 = 1.94985256. \]

Answer B.

**Exercise 6.2**
You are an investor purchasing the following bond:

- 8-year maturity.
- 4% coupon rate, coupons paid semiannually.
- Priced to yield 4% per year, compounded semiannually.
- Par value = 1000.
- Market value upon purchase = 1000.
- Noncallable bond.
- Macaulay duration = 6.8327 years.

One day after purchasing the bond, interest rates increase to a 6% nominal annual rate compounded semiannually and remain at this level until bond maturity. You suffer a capital loss on the bond and really do not like it. Assume bond coupons can be re-invested at a 6% nominal annual rate compounded semiannually. You decide to sell the bond immediately after the first coupon payment after which you will receive at least the
same yield as the original yield when you purchased the bond. When will you sell the bond?

A. Immediately after the thirteenth coupon is paid
B. Immediately after the fourteenth coupon is paid
C. Immediately after the fifteenth coupon is paid
D. Immediately after the sixteenth coupon is paid
E. It is impossible to recover your original yield

Solution.
This exercise illustrates the idea in the graph showing how investment professionals think about duration. The time of sale should be the original duration (i.e., duration at the time of purchase) of the bond. The Macaulay duration at purchase is 6.8327 years, and therefore the (modified/effective) duration is 
\[ \frac{6.8327}{1.02} = 6.5674. \]
This actually means that it will take exactly 6.5674 years to have the original yield of 4% nominal compounded semiannually (i.e., 4.04% annual effective) recovered after the capital loss due to rate increase. Because you must sell immediately after a payment of a coupon, you will wait until the fourteenth coupon. This is actually all you need to state to solve this problem.

Just as a comment, note that after the fourteenth coupon payment you will have the following:

- Market value of the bond at sale after the 14-th coupon:
  \[ 20a_{\overline{14}|3}\% + \frac{1000}{1.03^2} = \frac{1020}{1.03^2} + \frac{20}{1.03} = 961.45 + 19.42 = 980.87. \]

- In addition to that, the investor has accumulated 14 coupons of 20, at 3% per half year, valued at:
  \[ 20 \cdot s_{\overline{14}|3}\% = 20 \cdot \frac{1.03^{14} - 1}{0.03} = 341.73. \]

- Total value received after 14 payments = 1322.60. The annual rate of return \( r \), which gives that, is the solution of the equation
  \[ 1000(1 + r)^7 = 1322.60, \]
i.e., \( r = 4.08\% \). This is just slightly over 4.04% initial effective annual yield. Incidentally, after the thirteenth coupon, market value of the bond is:

\[ 20a_{\overline{13}|3}\% + \frac{1000}{1.03^3} = \frac{1020}{1.03^3} + \frac{20}{1.03^2} + \frac{20}{1.03} = 933.44 + 18.85 + 19.42 = 971.71. \]
The accumulated value of the coupons is
\[ 20 \cdot s_{\overline{13}|3}\% = 20 \cdot \frac{1.03^{13} - 1}{0.03} = 312.36. \]
Total value of the accumulated investment is
\[ 971.71 + 312.36 = 1284.07. \]
This corresponds to semiannual yield of 1.94% or effective annual yield of 3.92%, obviously less than 4.04%.
Answer B.
Exercise 6.3
You are managing a bond portfolio of $1,000,000. You decide that the Macaulay duration of your portfolio should be exactly 10. You have only two securities to choose from for your investments: a zero-coupon bond of maturity 5 years, and a continuous perpetuity paying at the rate of $1 per year. Current force of interest is 5%. How much will you invest in each of these securities in order to have the desired Macaulay duration?

A. $250,000 in the zero, $750,000 in perpetuity
B. $333,333 in the zero, $666,666 in perpetuity
C. $500,000 in the zero, $500,000 in perpetuity
D. $666,666 in the zero, $333,333 in perpetuity
E. $750,000 in the zero, $250,000 in perpetuity

Solution.
The Macaulay duration of the zero coupon bond will be exactly 5 years. The perpetuity is worth $1/δ if the force of interest is δ, and so its derivative with respect to δ equals −1/δ², and its Macaulay duration is

\[-\frac{1}{\delta} = \frac{1}{\delta'}\]

At the given force of interest, this means its duration is 20. If you invest a portion of your portfolio \( w \) in the zero-coupon bond and \( 1 - w \) in the perpetuity, Macaulay duration of the entire portfolio will be

\[w \cdot 5 + (1 - w) \cdot 20 = 10\]

Thus

\[5w - 20w = 10 - 20\]

so that \( 15w = 10 \), and \( w = \frac{2}{3} \). You should invest $666,666.66 in the 5-year zero coupon bond and $333,333.33 in the perpetuity.

Answer D.

7. Convexity, applications of duration and convexity, and use of nominal interest rates
Let \( P(i) \) be the price of a security considered as a function of interest rate, and \( P(\delta) \). The quantity:

\[C_M(P(\delta)) = \frac{1}{P(\delta)} \frac{d^2P(\delta)}{d\delta^2}\]

will be termed the Macaulay convexity (this concept is rarely used in other books, but please tolerate it, because it is useful) of this security. Note that for a security with deterministic discrete cash flows...
\[ P(\delta) = \sum_{t=0} A_t e^{-\delta t}, \]

and

\[ C_M(P(\delta)) = \frac{1}{P(\delta)} \sum_{t=0} t^2 \cdot A_t e^{-\delta t} = \sum_{t=0} w_t \cdot t^2, \]

where, as before, \( w_t = \frac{\text{PV}(A_t)}{P} \).

Because

\[ P(\delta) \cdot D_M(P(\delta)) = -P'(\delta) \]

is the monetary (dollar) duration of the security, we also have:

\[ C_M(P(\delta)) = -\frac{1}{P(\delta)} \frac{d}{d\delta} \left( P(\delta) \cdot D_M(P(\delta)) \right) = D_M^2(P(\delta)) - \frac{dD_M(P(\delta))}{d\delta}. \]

The quantity

\[ M_M^2(P(\delta)) = -\frac{dD_M(P(\delta))}{d\delta} = C_M(P(\delta)) - D_M^2(P(\delta)) \]

will be termed the \textit{Macaulay M-squared}. Because

\[ D_M(P(\delta)) = -\frac{d}{d\delta} \ln P(\delta), \]

we also see that

\[ M_M^2(P(\delta)) = \frac{d^2(\ln P(\delta))}{d\delta^2}. \]

For a security with discrete deterministic cash flows

\[ P(\delta) = \sum_{t=0} A_t e^{-\delta t} \]

and:

\[ M_M^2(P(\delta)) = \frac{1}{P(\delta)} \sum_{t=0} \left( t - D_M(P(\delta)) \right)^2 e^{-\delta t} A_t = \sum_{t=0} w_t \left( t - D_M(P(\delta)) \right)^2 \]

and

\[ \frac{dM_M^2(P(\delta))}{d\delta} = -\frac{1}{P(\delta)} \sum_{t=0} \left( t - D_M(P(\delta)) \right)^3 e^{-\delta t} A_t = -\sum_{t=0} w_t \left( t - D_M(P(\delta)) \right)^3. \]

The same concepts are developed for the relationships with respect to interest rate. The expression

\[ C(P(i)) = \frac{1}{P(i)} \frac{d^2 P(i)}{dt^2} \]

is the \textit{convexity} of a security, and

\[ M^2(P(i)) = \frac{d^2(\ln(P(i)))}{dt^2} = -\frac{dD(P(i))}{di} \]

is called the \textit{M-squared} of a security.

What is the practical interpretation of the concept of convexity? We did note that for a security with deterministic cash flows the Macaulay duration is the weighted average time to maturity. Consider such a deterministic security. Let us again use the notation:
\[ w_t = \frac{PV(A_t)}{P(i)} = \sum_{t>0} A_t e^{-\delta t}. \]

Now observe that:
\[ M_m^2 = C_m - D_m^2 = \sum_{t>0} w_t \cdot t^2 - D_m^2 = \sum_{t>0} w_t \cdot (t - D_m)^2. \]

This allows us for a relatively simple and quite intuitive interpretation of Macaulay duration, Macaulay convexity and Macaulay M-squared of a security. As we stated before, Macaulay duration is the expected time to cash flow with respect to the probability distribution whose probability function (or probability density function, in the case of continuous payments) is \( f_T(t) = w_t \). Macaulay convexity is the second moment of this random variable, and Macaulay M-squared is the variance of it. This means that Macaulay duration can be interpreted intuitively as the expected time till maturity of cash flows of a security, Macaulay M-squared is the measure of dispersion of the cash flows of the said security, and Macaulay convexity is a sum of Macaulay M-squared and the square of Macaulay duration. Note also that by the Chain Rule:
\[ \frac{dP}{di} \frac{d\delta}{dP} = \frac{d\ln(1+i)}{di} \frac{dP}{d\delta} = \frac{1}{1+i} \frac{dP}{d\delta}. \]

Thus taking a derivative with respect to \( i \) is the same as taking a derivative with respect to \( \delta \) and multiplying the result by \( \frac{1}{1+i} \). Also, taking a derivative with respect to \( \delta \) is the same as taking a derivative with respect to \( i \) and dividing the result by \( \frac{1}{1+i} \), i.e.,
\[ \frac{d^2P}{di^2} = \frac{d}{di} \left( \frac{dP}{di} \right) = \frac{d}{di} \left( \frac{1}{1+i} \frac{dP}{d\delta} \right) = \frac{1}{(1+i)^2} \frac{dP}{d\delta} + \frac{1}{1+i} \frac{d}{di} \left( \frac{dP}{d\delta} \right) = \frac{1}{(1+i)^2} \frac{dP}{d\delta} + \frac{1}{1+i} \frac{d^2P}{d\delta^2}. \]

This means that:
\[ C = \frac{1}{(1+i)^2} D_m + \frac{1}{(1+i)^2} C_m. \]

For \( M^2 = C - D^2 \), we can calculate
\[ M^2 = \frac{d^2(\ln P)}{di^2} = \frac{d}{di} \left( \frac{d\ln P}{di} \right) = \frac{d}{di} \left( \frac{1}{P} \frac{dP}{di} \right) = -\frac{1}{P^2} \frac{dP}{di} \frac{dP}{di} + \frac{1}{P} \frac{d^2P}{di^2} = \frac{1}{(1+i)^2} C_m + \frac{1}{(1+i)^2} D_m - \frac{1}{(1+i)^3} D_m^2 = \frac{1}{(1+i)^2} M_m^2 + \frac{1}{(1+i)^2} D_m. \]
For a security with discrete deterministic cash flows $A_t$ (at time $t$), and $P = \sum_{t \geq 0} PV(A_t)$, its monetary (dollar) convexity is:

$$\frac{1}{(1+i)^2} \sum_{t \geq 0} t(t+1) \cdot A_t (1+i)^{-t}.$$ 

Furthermore:

$$C = \frac{1}{(1+i)^2} \sum_{t \geq 0} w_t t(t+1) = \frac{1}{(1+i)^2} C_M + \frac{1}{(1+i)^2} D_M.$$ 

Let us also note that we have a simple expression for the Macaulay convexity of a single payment at time $t$: it is equal to $t^2$. Its convexity is

$$1 + \frac{i}{2} t + \frac{i}{2} t \geq 0 \sum t.$$ 

Furthermore:

$$C = 1 + \frac{i}{2} t + \frac{i}{2} t \geq 0 \sum t + 1 \sum t = 1 + \frac{i}{2} t + \frac{i}{2} t C_M + \frac{1}{2} i D_M.$$ 

Macaulay convexity of any security, whose price is treated a function of interest rate (or force of mortality) $P = P(i) = P(\delta)$, and $P$, $P_1$, and $P_2$ are prices of securities, expressed as functions of interest rate, such that $P(i) = P(\delta) = P_1(i) \pm P_2(i) = P_1(\delta) \pm P_2(\delta)$, then

$$\text{Convex}(P) = \frac{P_1}{P} \text{Convex}(P_1) \pm \frac{P_2}{P} \text{Convex}(P_2) = \frac{P_1}{P_1 \pm P_2} \text{Convex}(P_1) \pm \frac{P_2}{P_1 \pm P_2} \text{Convex}(P_2).$$

Because $M^2_M$ is the variance of the random variable with density $f_T(t) = w_t$, it, and $M^2$, increase with dispersion of cash flows.

If a security has embedded options (direct interest rate options, such as a prepayment option, or the option to default), then the only practical calculation of convexity is an approximation. Note that we have the following approximate identities implied by the Taylor series expansion:

$$P(i + \Delta i) - P(i) \approx \frac{dP}{di} \Delta i + \frac{1}{2} \frac{d^2P}{di^2} (\Delta i)^2,$$

$$P(i - \Delta i) - P(i) \approx \frac{dP}{di} (-\Delta i) + \frac{1}{2} \frac{d^2P}{di^2} (-\Delta i)^2,$$

and by adding the two we get

$$P(i - \Delta i) - 2P(i) + P(i + \Delta i) = \frac{d^2P}{di^2} (\Delta i)^2$$

or

$$C \approx \frac{d^2P}{di^2} \frac{1}{P} \frac{P(i - \Delta i) - 2P(i) + P(i + \Delta i)}{P(i)(\Delta i)^2},$$

which is a popular approximation used in the case of valuing interest sensitive cash flows.
There is one more approximation to note:
\[
\Delta P = P(i + \Delta i) - P(i) = \frac{dP}{di} \Delta i + \frac{1}{2} \frac{d^2P}{di^2} (\Delta i)^2,
\]
and this implies
\[
\frac{\Delta P}{P} \approx -\left( -\frac{dP}{P} \right) \Delta i + \frac{1}{2} \left( \frac{d^2P}{P} \right) (\Delta i)^2 = -D \cdot \Delta i + \frac{1}{2} \cdot C \cdot (\Delta i)^2.
\]

**Exercise**

Your company has a liability which calls for a single payment of $1000 exactly due in ten years, and it funds it with a one year zero-coupon bond with a price of $200, and a twenty year zero coupon bond with a price of $800. Current continuously compounded interest rate (i.e., force of interest) is 4% and the yield curve is flat (interest rates for all maturities are the same). Calculate the Macaulay duration and Macaulay convexity of your company’s surplus.

Solution.
The surplus is defined as assets minus liabilities. Present value of the liability is
\[1000e^{-10 \cdot 0.04} = 670.32.\]
Therefore, this company’s surplus equals:
\[1000 - 670.32 = 329.68.\]
Macaulay duration of the surplus is the weighted average of the durations of individual cash flows:
\[
\frac{200}{329.68} \cdot 1 - \frac{670.32}{329.68} \cdot 10 + \frac{800}{329.68} \cdot 20 = 28.8061.
\]

Macaulay convexity of an individual cash flow is the square of its time to payment.
Macaulay convexity of the surplus is the weighted average of individual cash flows convexities:
\[
\frac{200}{329.68} \cdot 1^2 - \frac{670.32}{329.68} \cdot 10^2 + \frac{800}{329.68} \cdot 20^2 = 767.9204.
\]

Duration and convexity be calculated with respect to nominal interest rates, such as \(i^{(2)}, i^{(3)}, \ldots, i^{(m)}\)? Yes, and the most natural choice is \(i^{(2)}\), because semiannual yields are most common for bonds issued in the United States (although not common at all outside of the United States). Recall that
\[
\left( 1 + \frac{i^{(m)}}{m} \right)^m = 1 + i = e^{\delta}.
\]
Therefore
\[
\frac{di}{di^{(m)}} = m \cdot \frac{1}{m} \left( 1 + \frac{i^{(m)}}{m} \right)^{m-1} = \frac{1 + i}{i^{(m)}}
\]
and
\[
\frac{d\delta}{di^{(m)}} = \frac{d\delta}{di} \cdot \frac{di}{di^{(m)}} = \frac{1}{1 + \frac{i^{(m)}}{m}}.
\]

Combining those identities, we get:
\[
\frac{dP}{di^{(m)}} = \frac{dP}{di} \cdot \frac{di}{di^{(m)}} = \frac{1 + i}{i^{(m)}} \cdot \frac{dP}{di},
\]
\[
\frac{dP}{di^{(m)}} = \frac{dP}{d\delta} \cdot \frac{d\delta}{di^{(m)}} = \frac{1}{1 + \frac{i^{(m)}}{d\delta}} \cdot \frac{dP}{d\delta}.
\]

Based on these calculations, we can derive the duration with respect to \(i^{(m)}\) as
\[
D^{(m)} = \frac{1 + i}{i^{(m)}} D = \frac{1}{1 + \frac{i^{(m)}}{m}} D_M
\]
and the duration with respect to \(\frac{i^{(m)}}{m}\) as \(m \cdot D^{(m)}\). What about convexity measure with respect to \(i^{(m)}\)? It is virtually impossible that an exam question will be asked about it, but just in case, have a look at this:
\[
\frac{d^2P}{d\left(i^{(m)}\right)^2} = \frac{d}{di^{(m)}} \left(\frac{dP}{di^{(m)}}\right) = \frac{d}{d\delta} \left(\frac{1}{1 + \frac{i^{(m)}}{d\delta}} \cdot \frac{dP}{d\delta}\right) = \frac{1}{\left(1 + \frac{i^{(m)}}{m}\right)^2} \frac{d^2P}{d\delta^2} + \frac{1}{\left(1 + \frac{i^{(m)}}{m}\right)^2} \frac{1}{m} \left(-\frac{dP}{d\delta}\right).
\]

Therefore, convexity with respect to \(i^{(m)}\) equals:
\[
C^{(m)} = \frac{1}{\left(1 + \frac{i^{(m)}}{m}\right)^2} C_M + \frac{1}{\left(1 + \frac{i^{(m)}}{m}\right)^2} \frac{1}{m} D_M.
\]

But there is a simpler way around dealing with nominal interest rates. Any interest rate is always given over a specific period of time, duration is expressed in units of time, and the specific unit of time used for duration is that unit over which the interest rate is given. For an annual interest rate, duration is expressed in years. But if we have to deal with a nominal interest rate, we can just switch to the effective rate over a fraction of a year, and express time in those fractions of year. For example, if a nominal interest rate is compounded quarterly, then we can switch to the effective quarterly rate, and express duration in quarters, instead of years. For convexity, the standard unit is year-squared. But if we have an effective quarterly rate, convexity will be expressed in quarters-squared. Because there are four quarters in a year, there are sixteen quarters-squared in a year-squared.
Fisher-Weil duration and convexity

As defined above, both duration and convexity are only applicable for one interest rate, i.e., a flat yield curve. This is a common complaint about standard definition of duration and convexity. One possible way to address it is by using the whole yield curve for discounting.

Let $P(0,t)$ be the time 0 price of a zero coupon bond maturing at $1$ face at time $t$. The following are natural generalizations of Macaulay concepts:

$$
\sum_{t \geq 0} t \cdot (CF_t) \cdot P(0,t) \quad \text{Fisher-Weil monetary duration,}
$$

$$
\sum_{t \geq 0} t^2 \cdot (CF_t) \cdot P(0,t) \quad \text{Fisher-Weil monetary convexity,}
$$

$$
\sum_{t \geq 0} (CF_t) \cdot P(0,t) \quad \text{Fisher-Weil duration,}
$$

$$
\sum_{t \geq 0} t^2 \cdot (CF_t) \cdot P(0,t) \quad \text{Fisher-Weil convexity.}
$$

Exercise.

Consider a three-year 10% annual coupon bond. The current spot yield curve on a continuously compounded basis is (rates are annual forces of interest for one-year, two-year, and three-year spot rates):

$$
\delta_1 = 10\%, \delta_2 = 11\%, \delta_3 = 12\%.
$$

Calculate the Fisher-Weil convexity of the bond.

Solution.

Without loss of generality, we can assume $100$ face amount of the bond. We have:

$$
C_{F-W} = \frac{\sum_{i=1}^{3} t^2 \cdot CF_i \cdot P(0,t)}{\sum_{i=1}^{3} CF_i \cdot P(0,t)} = \frac{1^2 \cdot 10 \cdot e^{-0.10} + 2^2 \cdot 10 \cdot e^{-0.11^2} + 3^2 \cdot 110 \cdot e^{-0.12^3}}{10 \cdot e^{-0.10} + 10 \cdot e^{-0.11^2} + 110 \cdot e^{-0.12^3}} = 7.801.
$$

May 2000 SOA Course 6 Examination, Multiple Choice, Problem No. 3

You are given the following with respect to a bond with semi-annual coupon payments priced to yield 8% (annual nominal compounded semi-annually):

<table>
<thead>
<tr>
<th>Semi-Annual Period (t)</th>
<th>Payment (t)</th>
<th>Present Value of Payment at Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.00</td>
<td>2.89</td>
</tr>
<tr>
<td>2</td>
<td>3.00</td>
<td>2.78</td>
</tr>
</tbody>
</table>

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Calculate the convexity the bond with respect to the nominal annual interest rate \( i^{(2)} \) compounded semiannually.

A. 1.29  
B. 1.58  
C. 8.78  
D. 9.50  
E. 17.56

Solution.
Note first that the sum of present values is 95.14. The convexity with respect to the force of interest is:

\[
\begin{align*}
0.5^2 \cdot \frac{2.89}{95.14} + 1^2 \cdot \frac{2.78}{95.14} + 1.5^2 \cdot \frac{2.67}{95.14} &+ 2^2 \cdot \frac{2.57}{95.14} + \\
+ 2.5^2 \cdot \frac{2.47}{95.14} + 3^2 \cdot \frac{81.76}{95.14} &= 8.10455644. \\
\end{align*}
\]

This is closest to C. The Macaulay duration is

\[
\begin{align*}
0.5 \cdot \frac{2.89}{95.14} + 1 \cdot \frac{2.78}{95.14} + 1.5 \cdot \frac{2.67}{95.14} &+ 2 \cdot \frac{2.57}{95.14} + \\
+ 2.5 \cdot \frac{2.47}{95.14} + 3 \cdot \frac{81.76}{95.14} &= 2.78352954. \\
\end{align*}
\]

Convexity with respect to the interest rate \( i \) is:

\[
C = \frac{1}{(1+i)^2} \cdot C_M + \frac{1}{(1+i)^2} \cdot D_M = \\
= \frac{1}{1.04^2} \cdot 8.10455644 + \frac{1}{1.04^2} \cdot 2.78352954 = 9.30718153. \\
\]

This is closest to D. What is the value of convexity with respect to \( i^{(2)} \)? Convexity with respect to nominal annual interest rate \( i^{(m)} \) is

\[
C^{(m)} = \frac{1}{(1+i^{(m)})^2} \cdot C_M + \frac{1}{(1+i^{(m)})^2} \cdot \frac{1}{m} D_M. \\
\]

Therefore,

\[
C^{(2)} = \frac{1}{(1+i^{(2)})^2} \cdot C_M + \frac{1}{(1+i^{(2)})^2} \cdot \frac{1}{2} D_M = \\
= \frac{1}{1.04^2} \cdot 8.10455644 + \frac{1}{1.04^2} \cdot \frac{1}{2} \cdot 2.78352954 = 8.77988278. \\
\]

This is exactly answer C. You actually also get answer C, if you treat fractional times as whole numbers, i.e., count time in half years, and use the basic formula for convexity for fixed cash flows.
\[ C = \frac{1}{(1+i)^2} \sum_{t \geq 0} w_t (t+1), \]

so that
\[
\begin{align*}
1 \cdot 2 & \cdot \frac{2.89}{1.04^2} + 2 \cdot 3 \cdot \frac{2.78}{104^2} + 3 \cdot 4 \cdot \frac{2.67}{1.04^2} + 4 \cdot 5 \cdot \frac{2.57}{1.04^2} + \\
5 \cdot 6 & \cdot \frac{2.47}{1.04^2} + 6 \cdot 7 \cdot \frac{81.76}{1.04^2} + 7 \cdot 81.76 & \approx 35.1195311.
\end{align*}
\]

The convexity measure in years-squared is therefore
\[
\frac{35.1195311}{4} \approx 8.77988278.
\]

Division by four is caused by the fact that there are four half-years squared in one year squared. Or, you can simply observe that the calculation produces convexity with respect to
\[
j = \frac{i^{(2)}}{2},
\]

and that
\[
\frac{d^2 P}{d(i^{(2)})^2} = \frac{1}{4} \frac{d^2 P}{dj^2}.
\]

Answer C.

8. Classical Immunization

Assume that a financial intermediary has both assets and liabilities, whose values both depend on the interest rate, and are denoted by \( A(i) \) and \( L(i) \), respectively. Then the difference of the two:
\[
S(i) = A(i) - L(i),
\]
termed the surplus, or capital, of the enterprise, is the subject of intense regulatory scrutiny, and management’s interest. While in practice its value may be established not by the market, but by the regulatory or accounting principles, it is important that managers of a financial intermediary understand the relationship of surplus value (market value) to interest rate. Redington (in 1952) proposed integrated treatment of assets and liabilities through the study of the surplus as a function of interest rates. If the objective of the financial intermediary is to prevent the surplus value from changing when the interest rate changes, one possible approach is to seek the change in the value of \( S \) to be close to zero when the interest rate changes slightly, i.e., to have
\[
\Delta S \approx 0 \text{ for } \Delta i = 0.
\]
This implies that
\[
\frac{dS}{di} = \frac{dA}{di} - \frac{dL}{di} = 0,
\]
i.e., the nominal duration of assets must be set equal to the nominal duration of liabilities. If, additionally, we want to assure ourselves that when the interest rate changes, the value of surplus actually increases, we want \( \frac{d^2 S}{di^2} > 0 \), so that the graph of the surplus as a
function of the interest rate is convex. This can be achieved by having assets of greater nominal convexity than that of liabilities. Thus, this form of immunization can be summarized as follows:

- To protect the absolute surplus level, set \( \frac{dA}{di} = \frac{dL}{di} \), that is, monetary (dollar) duration of assets equal to the dollar duration of liabilities, and \( \frac{d^2A}{di^2} > \frac{d^2L}{di^2} \), that is, choose assets with more nominal convexity than the nominal convexity of liabilities.

But an intermediary may be more concerned with protecting its ratio of assets to liabilities than the absolute surplus level. This may be a result of the common regulatory concern with capital ratio (i.e., ratio of surplus to assets), or management’s desire to control risk by monitoring the capital ratio. In this case, the firm would be interested in setting the derivative of the ratio of assets and liabilities with respect to the interest rate to zero, while keeping the second derivative positive. Since the natural logarithm is a strictly increasing function, this approach is equivalent to keeping the derivative of the natural logarithm of the ratio of assets to liabilities equal to zero, while keeping the second derivative positive. The resulting approach is the most common form of classical immunization:

- To protect the surplus ratio level, set
  \[
  \frac{d \ln(A(i))}{di} = \frac{d \ln(L(i))}{di},
  \]
  or, equivalently, set duration of assets equal to the duration of liabilities, and
  \[
  \frac{d^2 \ln(A(i))}{di^2} > \frac{d^2 \ln(L(i))}{di^2},
  \]
  that is, choose assets with more \( M^2 \) (dispersion) than that of liabilities. But when durations of assets and liabilities are equal, greater \( M^2 \) is equivalent to greater convexity, so that one can also set this condition as convexity of assets exceeding convexity of liabilities. This is the approach in standard applications of immunization. We should note that immunization has many critics, and the most commonly quoted list of problems with immunization is as follows:

- Immunization assumes one interest rate, i.e., flat yield curve, which only moves in parallel shifts.
- Immunization assumes only instantaneous infinitely small change in the yield curve, and of course this is not the kind of changes an intermediary usually experiences.
- Immunization requires continuous costly rebalancing, because of changes in assets and liabilities values, durations, and convexities, with changes in the market level of interest rates, and with the passage of time.

**Exercise 8.1**
You are given the following information concerning an immunization strategy considered by an insurance company:
- Initial portfolio value: 10,000,000.
- Minimum return required: 4.0%. If the company earns this return, current assets will pay off their liability in full at the end of the investment horizon.
- Number of years in investment horizon: 10.
- Immunized yield available: 8.0%.

Assuming that immunization strategy can work perfectly if implemented, calculate the difference between the initial portfolio value and the required assets for immunization at inception.

Solution.
Terminal value of assets required is
\[ 10000000 \cdot (1 + 0.04)^{10} = 14802442.80. \]
Since the immunization strategy is assumed to work perfectly, the insurance company can get 8% return, and the present value of assets needed to pay the liability is
\[ \frac{14,802,442.80}{1.08^{10}} = 6,856,935.14. \]
The difference between the initial portfolio value and this amount is:
\[ 10,000,000 - 6,856,935.14 = 3,143,604.86. \]

9. The yield curve and multivariate immunization

Until now, we have assumed that we can use the same interest rate for discounting cash flows for all maturities. In reality, the rate used for discounting cash flows for various maturities differ. This can be observed most directly by comparing interest rates for the pure discount bonds, also known as zero coupon bonds, i.e., bonds that make only one payment at maturity, and no intermediate coupon payments. The yield curve, or term structure of interest rates is the pattern of interest rates for discounting cash flows of different maturities. The specific functional relationship between the time of maturity and the corresponding interest rate is usually called the yield curve, especially when represented graphically, while term structure of interest rates is the general description of the phenomenon of rates varying for different maturities. Longer-term bonds usually offer higher yields. Such a pattern of interest rates is termed an upward sloping yield curve. If interest rates are the same for all maturities, we call this a flat yield curve. Finally, a rare, but sometimes occurring, situation when longer-term maturity interest rates are lower than shorter-term rates, is an inverted yield curve.

In fact, there are three ways to define the yield curve (and term structure of interest rates). The first one assigns to each maturity date the coupon rate of a bond of that maturity trading at par (typically a newly issued bond). This is termed the bond yield curve. The second approach assigns to each maturity date the interest rate on a zero-coupon bond of that maturity. This yield curve is called the spot curve, and the interest rates given by it are called spot rates. The third approach uses short-term interest rates in the future periods of time implied by current bond spot rates. Let us explain this concept. A short-term interest rate, or short rate, is an interest rate applicable for a short period of time, up to one year, including the possibility of an instantaneous rate over the next infinitesimal period of time. If we use the one-year rate as the short rate for the purpose of deriving forward rates, we have the following relationship
\[ (1 + s_n)^n = (1 + f_1)(1 + f_2)\ldots(1 + f_n), \]
where \( s_n \) is the spot rate for maturity \( n,n=1,2,\ldots \) and \( f_i \) is the forward rate from time \( i - 1 \) till time \( i \), \( i = 1,2,\ldots \). We also have:
\[ 1 + f_n = \frac{(1 + s_n)^n}{(1 + s_{n-1})^{n-1}}. \]

The yield curve can also be studied for the continuously compounded interest rate, i.e., for the force of interest. In this case the force of interest \( \delta = \delta(t) = \delta_t \) is a function of time. The distinction between the spot rate and the forward rate is best explained by presenting their mathematical relationship. If \( \delta_t \) the spot force of interest for time \( t \) and \( \varphi_t \) is the forward force of interest at time \( t \), then the accumulated value at time \( t \) of a monetary unit invested at time 0 is:

\[
\left( e^{\delta} \right)^t = e^{\int_0^t \varphi_s \, ds}.
\]

Therefore we have

\[ t \delta_t = \int_0^t \varphi_s \, ds, \]

or

\[ \delta_t = \frac{1}{t} \int_0^t \varphi_s \, ds, \]

i.e., the spot rate for time \( t \) is the mean value of the forward rates between times 0 and \( t \). Note also that by the Fundamental Theorem of Calculus:

\[ \varphi_t = t \frac{d \delta_t}{dt} + \delta_t. \]

This shows us that if \( \frac{d \delta_t}{dt} \) is positive then \( \varphi_t > \delta_t \), and if \( \frac{d \delta_t}{dt} < 0 \) then \( \varphi_t < \delta_t \).

Exercise 9.1
You are given the following information about risk-free zero coupon bonds:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Price (per 1,000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>962</td>
</tr>
<tr>
<td>2 year</td>
<td>916</td>
</tr>
<tr>
<td>3 year</td>
<td>864</td>
</tr>
</tbody>
</table>

Calculate 1, 2, and 3-year spot interest rates.

Solution.

1 year spot rate = \( \frac{1000}{962} - 1 = 3.9501\% \), because \( 1000 = 962 \cdot (1 + s_1) \).

2 year spot rate = \( \left( \frac{1000}{916} \right)^{0.5} - 1 = 4.48459\% \), because \( 1000 = 916 \cdot (1 + s_2)^2 \).

3 year spot rate = \( \left( \frac{1000}{864} \right)^{1/3} - 1 = 4.99342\% \), because \( 1000 = 864 \cdot (1 + s_3)^3 \).
**Exercise 9.2**
For the same data as in the previous exercise, calculate 1, 2, and 3-year forward rates (i.e., forward rates for one-year rates from times 0 to 1, 1 to 2, and 2 to 3).

Solution.
First, \( (1 + s_1) = (1 + f_1) \) so that \( f_1 = 3.9501\% \). Then
\[
(1 + s_2)^2 = \frac{1000}{916} = (1 + f_1)(1 + f_2) = \frac{1000}{962}(1 + f_2),
\]
so that \( 1 + f_2 = \frac{962}{916}, \) and \( f_2 = 5.02183\% \). Finally,
\[
(1 + s_3)^3 = \frac{1000}{864} = (1 + f_1)(1 + f_2)(1 + f_3) = \frac{1000}{962} \cdot \frac{962}{916}(1 + f_3),
\]
so that \( 1 + f_3 = \frac{916}{864}, \) and \( f_3 = 6.01852\% \).

**Exercise 9.3**
You are given that the current forward rate from time 0 to time 1 is 5\%, from time 1 to time 2 it is 6\%, and from time 2 to time 3 it is 7\%. You purchase a three-year coupon bond paying an annual coupon of 60, with the payment of 1000 principal at maturity. If in one year the yield curve is flat at 7\% for all maturities, what is the total return of this bond for the year?

Solution. 
The current price of this bond is
\[
\frac{60}{1.05} + \frac{60}{1.05 \cdot 1.06} + \frac{1060}{1.05 \cdot 1.06 \cdot 1.07} = 57.14 + 53.91 + 890.08 = 1001.13.
\]
At the end of the year, the bond will pay a coupon of 60, and its price will be
\[
\frac{60}{1.07^2} + \frac{1060}{1.07} = 56.07 + 925.85 = 981.92.
\]
Thus the 1001.13 investment will provide the total payoff of 1041.92 = 981.92 + 60.00, for a rate of return of
\[
\frac{1041.92}{1001.13} - 1 = 4.07\%.
\]

To address some weaknesses of the classical immunization, Ho (in 1990) and Reitano (in 1991) developed a multivariate generalization of duration and convexity. They replaced the single interest rate parameter \( i \) by a yield curve vector \( \tilde{i} = (i_1, \ldots, i_n) \), where the coordinates of the yield curve vector correspond to certain set of “key” rates (e.g., 0.25, 0.5, 1, 2, 3, 4, 5, 7, 10, 20 and 30 year). The price function is then \( P(i_1, \ldots, i_n) \), and instead of analyzing derivatives with respect to one interest rate variable, one could use multivariate calculus tools to study this price function. There is one objection that could be raised with respect to this approach immediately. When analyzing a function of two
variables \( f(x,y) \) we implicitly assume that the variables \( x \) and \( y \) are independent of each other, that is, that each of them has its derivative with respect to the other equal to zero. This is definitely not the case when various maturity interest rates are considered. Nevertheless, one can study such multivariate models for the purpose of better understanding of their properties.

The negative partial logarithmic derivatives of \( P(i_1, \ldots, i_n) \) are then termed partial durations (Reitano), or key-rate durations (Ho). The total duration vector is:

\[
- \frac{P'(i_1, \ldots, i_n)}{P(i_1, \ldots, i_n)} = - \frac{1}{P(i_1, \ldots, i_n)} \left( \frac{\partial P}{\partial i_1}, \ldots, \frac{\partial P}{\partial i_n} \right).
\]

One can also introduce the standard notion of directional derivative of \( P(i_1, \ldots, i_n) \) in the direction of a vector \( \vec{v} = (v_1, \ldots, v_n) \):

\[
P'_\vec{v}(i_1, \ldots, i_n) = \vec{v} \cdot \left( \frac{\partial P}{\partial i_1}, \ldots, \frac{\partial P}{\partial i_n} \right).
\]

The dot refers to the dot product of the vectors. The second derivative matrix can also be used to define the total convexity:

\[
- \frac{P''(i_1, \ldots, i_n)}{P(i_1, \ldots, i_n)} = - \frac{1}{P(i_1, \ldots, i_n)} \left[ \frac{\partial^2 P}{\partial i_k \partial i_l} \right]_{1 \leq k, l \leq n}.
\]

One can now view the surplus of an insurance firm as a function of the set of key interest rates chosen \( S = S(i_1, \ldots, i_n) = S(\vec{i}) \), and use multivariate calculus for two immunization algorithms, directly analogous to the one-dimensional case:

- To protect the absolute surplus level, set the first derivative (gradient) equal to zero \( S'(\vec{i}) = 0 \), (note that \( 0 \) is the zero vector, with all its components being zero) and make the second derivative matrix positive definite.
- To protect the relative surplus level (i.e., surplus ratio), set

\[
\frac{A'(\vec{i})}{A(\vec{i})} = - \frac{L'(\vec{i})}{L(\vec{i})},
\]

and make the total convexity matrix positive definite; with the symbols \( A, L \) referring to assets and liabilities, respectively.

**Exercise 9.4**

You manage a bond portfolio for a life insurance company. You have observed that when the one-year spot interest rate rose from 3.00% to 3.20% and the ten-year spot rate rose from 3.50% to 3.75%, your portfolio lost 1.00% of its value. On the other hand, when the one-year spot rate fell from 3.20% to 2.90% and the ten-year spot rate fell from 3.75% to 3.60%, your portfolio gained 0.75% of its value. Estimate the one-year and ten-year key-rate durations (also known as partial durations) of your portfolio.

**Solution.**
Denote the key-rate durations with respect to one year and ten years, respectively, by $D_1$, $D_{10}$, and the respective spot interest rates by $i_1$, $i_{10}$. Then we have the following approximation derived from the definition of partial durations:

$$\frac{\Delta P}{P} \approx -D_1 \cdot \Delta i_1 - D_{10} \cdot \Delta i_{10}.$$ 

Substituting the data given we get the system of two linear equations:

$$-1.00\% = -D_1 \cdot 0.20\% - D_{10} \cdot 0.25\%,$$

$$0.75\% = -D_1 \cdot (-0.30\%) - D_{10} \cdot (-0.15\%).$$

This system of linear equations solves to $D_1 = \frac{5}{6}$, $D_{10} = \frac{10}{3}$.

10. Common stocks, preferred stocks, and their durations; short sales

Common stock is a financial asset, which represents partial ownership in a business enterprise, and thus gives its owner the right to a share in that enterprise’s profits. A portion of the profits paid out in cash to the owners is called a dividend. Dividends may also be paid from any cash held by the enterprise, not necessarily from its profits. The value of a share of common stock in the enterprise under consideration can be established by the market, if there is an active market for the shares (such as a stock exchange, e.g., New York Stock Exchange), or it must be estimated. When the value is estimated, we generally assume that the value equals one of these three:

- What you can sell the share for right now,
- Present value of dividends between now and some specific point in the future when you can sell your share, together with the present value of the share’s future sale price,
- Present value of dividends paid into the infinite future time horizon.

For a common stock, its dividend generally increases over time. But there is another type of fractional ownership in a business enterprise, the preferred stock, which is designed to typically pay a fixed dividend in perpetuity. Of course you know that you can establish value of such a security by calculating the present value of a perpetuity. Dividends are typically paid at the end of the year, thus a typical preferred stock is a perpetuity-immediate. Common stock can be viewed as an increasing perpetuity immediate, and usually the dividends are assumed to increase in a geometric progression.

In this analysis, we assume that the value of a stock is the present value of its dividends. If the dividend yield is constant, and dividends $D$ are constant and paid in perpetuity at the end of each year (which is the basic model for a preferred stock), then the price of the stock is $\frac{D}{i}$, and its duration is $-\frac{i}{D} \cdot \frac{d}{di} \left(\frac{D}{i}\right) = -i \cdot \frac{1}{d} \left(i^{-1}\right) = i \cdot i^{-2} = \frac{1}{i}$. 

Note that in particular, the duration of a unit perpetuity immediate is its price. The Macaulay duration of a level perpetuity immediate is $\frac{1}{i}$, and the price of a unit perpetuity due is $\frac{1}{vi}$, i.e., it equals to the price of a unit perpetuity due. The Macaulay duration of a unit perpetuity due, on the other hand, is simply the Macaulay duration of a unit perpetuity immediate reduced by one, as all cash flows come exactly one year earlier, i.e., it is
\[
\frac{1}{d} - 1 = \frac{1}{v} - \frac{v}{v} = \frac{1 - d}{v} = \frac{v}{i} = 1.
\]

And the duration of a unit perpetuity due is therefore
\[
\frac{1}{i(1+i)} = \frac{v}{i}.
\]

Now assume that a stock pays a dividend of \(D_1\) at time 1 (and, in general it pays a dividend of \(D_t\) at time \(t\)), and that its dividend grows at a constant rate of \(g\). Assume also that the annual yield rate used for the discounting of its cash flows, i.e., dividends, is \(i\). We must have \(g < i\) for this model to make sense. The price of the stock in this model is the present value of the dividends paid in perpetuity, i.e.,
\[
P_0 = \frac{D_1}{1+i} + \frac{D_1(1+g)}{(1+i)^2} + \frac{D_1(1+g)^2}{(1+i)^3} + \ldots = \frac{D_1}{1+i} \sum_{k=0}^{\infty} \left( \frac{1+g}{1+i} \right)^k =
\]
\[
= \frac{D_1}{1+i} \cdot \frac{1}{1-\frac{1+g}{1+i}} = \frac{D_1}{i-g}.
\]

Therefore, the duration of the stock can be calculated as:
\[
-\frac{1}{P_0} \cdot \frac{dP_0}{di} = -\frac{1}{P_0} \cdot \frac{d}{di} \left( \frac{D_1}{i-g} \right) = -\frac{D_1}{P_0} \cdot \frac{d}{di} \left( (i-g)^{-1} \right) =
\]
\[
= -(i-g) \cdot (-1)(i-g)^{-2} = \frac{1}{i-g}.
\]

You should also note that in this model, the dividend yield of the stock is
\[
\frac{D_1}{P_0} = i - g,
\]
and its duration is simply the reciprocal of the dividend yield.

A short sale is a sale of a security, which is borrowed (from a broker), with that security replaced later by a purchase (covering of a short). Short sale requires a deposit (margin deposit) of a certain amount of money (typically 50% of the amount received from the sale of the security). This margin deposit may, or may not, earn interest with the broker. The money received from the sale generally does not earn interest, and is held with the broker until the short is covered. A short sale creates new supply of the security shorted, so if the shorted security pays income (e.g., dividends), that income must be produced by the short seller, in order for the holder of the newly created security to receive such income.

An issue that arises in short sales is the calculation of the rate of return received by the short seller. In such a calculation, we should note that the rate of return is determined by the initial cash outlay of the investor, and the final cash flow received at the end, when the short is covered, as well as possible intermediate cash flows of dividends paid by the investor and interest received.

General formula for the effective yield earned over the period of investment is:
\[ i = \frac{P + I - D}{M}, \]

where the symbols have the following meanings:

- \( M \) = margin requirement (initial cash outlay by the short seller),
- \( D \) = dividends paid by the short seller to the newly created security’s owner,
- \( I \) = interest earned by the short seller on the margin deposit (assuming there is no interest earned on the cash received from the initial short sale, if there is such interest, then it must be added here),
- \( P \) = profit on the short sale transaction, i.e., Sale Proceeds – Short Covering Repurchase Cost.

**May 2003 SOA/CAS Course 2 Examination, Problem No. 36**

Eric and Jason each sell a different stock short at the beginning of the year for a price of 800. The margin requirement for each investor is 50% and each will earn an annual effective interest rate of 8% on his margin account. Each stock pays a dividend of 16 at the end of the year. Immediately thereafter, Eric buys back his stock at a price of \((800 - 2X)\), and Jason buys back his stock at a price of \((800 + X)\). Eric’s annual effective yield, \( i \), on the short sale is twice Jason’s annual effective yield. Calculate \( i \).

A. 40%  B. 6%  C. 8%  D. 10%  E. 12%

Solution.

We have:

- Eric’s cash inflow = \(2X + 32 - 16 = 16 + 2X\)
- Jason’s cash inflow = \(-X + 32 - 16 = 16 - X\)

Margin deposit (initial cash outlay) = 400 for both Eric and Jason. Thus we have, by comparing yields

\[
\frac{16 + 2X}{400} = 2 \left( \frac{16 - X}{400} \right),
\]

which implies that \( X = 4 \). Eric’s yield is

\[
i = \frac{16 + 2 \cdot 4}{400} = 6%.
\]

Answer B.

**11. Overview of financial risk management**

What is the purpose of the existence of financial intermediaries, i.e., banks, insurance companies, finance companies, and investment companies? The old story explaining the nature of the work of a banker is: “Borrow at 3%, lend at 5%, and be at a golf course at 4.”

This is nice work if you can get it … but:

- On one hand, we do not believe this is possible,
- On the other hand, we do believe that banks, insurance companies, and financial intermediaries, would very much like to be in such a position, and try to achieve it any time they can.
Why do we not believe this is possible? Because we generally do not believe that the entire business of the bank, or any other financial intermediary can be rooted in arbitrage. Arbitrage is defined creation of a portfolio without an outlay of capital and without risk, yet earning income (with positive probability, so that there can be outcomes where there is no income, but there is no loss ever). We tend to expect markets to be arbitrage-free, as arbitrage opportunities would be exploited until they disappear. In the case of the banker, we simply do not expect the customers of the bank to ignore the existence of the investment opportunities that allow the bank to earn the 5%.

Thus, if financial intermediaries do not do arbitrage, what do they do? In a world without financial intermediaries, the national economy’s flow of funds would concentrate on the exchange between the household sector, which contains net savers, or better yet: net purchasers of securities, and the production (i.e., business) sector (corporations and other businesses), which contains net borrowers, or net providers of securities. However, because of monitoring costs, liquidity costs, price risk, and similar financial reasons, the average household saver may view investment in securities provided by the business sector as unattractive. Thus, financial intermediaries step in-between to “grease the wheels of commerce.” They provide products (those are financial assets, even though not traded) which are indeed needed by the household sector (thus assuming a short position in those securities), and use the funds to purchase securities supplied by the corporate sector (therefore assuming a long position in those securities), and other net issuers of securities (governments, government-sponsored enterprises, and even consumers themselves, when they borrow). There are also other types of situations where this short/long portfolio is automatically created by intermediation. For example, in payment facilitation, where the intermediary is short accounts receivable and long accounts payable. We submit that creation of such a short/long portfolio is the essence of the intermediation business, and, therefore, the essence of ALM. Also, the protection of the resulting link between the savers and producers is one of the major missions of financial intermediaries regulation.

What financial intermediaries do is often described as the “spread business.” This, of course, is a very traditional perspective. It is now widely acknowledged that financial intermediaries write options included in their products and their portfolios, in general. What we would like to point out is that coupon-clipping “spread” description addresses only the manner in which banks, insurance companies, or investment companies are paid, while the option-writing, and derivatives creation, is a much wider, and a much more accurate description of the complex nature of intermediaries’ portfolios. The business is built on assuming a long position in securities created by the business sector and a short position in securities issued to the savers (i.e., bank accounts, insurance policies, investment accounts, and so forth). Thus financial intermediaries effectively restructure cash flows provided by the securities of the business sector into the cash flows demanded by the household sector. A security created out of cash flows of another security is termed a derivative security. The derivative created by a financial intermediary, however, must be understood in a much broader sense than commonly used, i.e., not just restricted to options, or futures. In the popular press, it is often assumed that derivatives increase risk. One of the most common features of derivatives issued by financial intermediaries is that they actually reduce risk faced by their customers.
This perspective implies that financial intermediaries always face financial risks, i.e., risks caused by changing levels of market variables, such as interest rates, credit risk, changing values of share prices, liquidity, etc. What are those risks? A committee appointed by the Society of Actuaries identified the following key risks faced by life insurance companies:

- C-1 risk: credit risk, and risk of changing level of share prices,
- C-2 risk: risk of improper pricing of risk insured,
- C-3 risk: risk of divergence of values of assets and liabilities of the firm, especially due to changing levels of interest rates,
- C-4 risk: other business, legal, government and regulatory risk.

Are these C-1, C-2, C-3, and C-4 risks the only ones that financial intermediaries face? While this is a pretty general classification, we also have these risks:

- Currency risk,
- Liquidity risk,
- Derivative specific risks,
- Catastrophic risks.

Financial intermediaries, as well as their government regulators, respond to the risks that face them. What are those responses?

- Regulation,
- Risk-based capital requirements, including Value-at-Risk,
- Rating agencies’ credit ratings (Moody’s, Standard and Poor’s, Fitch),
- Asset-liability management, risk management, enterprise risk management.

The key reason for regulation of financial intermediaries is that their insolvencies do not just result in misfortune for their owners, and their employees, but that such insolvencies bring about severe negative externalities to the entire economy, possibly even eliminating all of their financial resources. Governments do, however, typically provide some form of insurance of value of certain securities created for customers. For example, in the United States:

- Bank deposits are insured up to $100,000 by the Federal Deposit Insurance Corporation ($250,000 till 2013).
- Insurance regulators (not done by the federal government, but by state insurance commissioners) provide state guaranty funds for value of insurance contracts, with varying amounts of maximum insurance coverage.
- Values of investment accounts with brokerage firms are insured by the Securities Insurance Protection Corporation, generally up to $100,000, ($250,000 till 2013).

Those regulators also require that the financial intermediaries to hold a certain amount of funds in excess of the amount held for the customers, i.e., certain amount of capital. That required capital is generally dependent on the amount of risk assumed by the financial firm. The more risk a firm assumes, the more capital it must hold. Those requirements are called Risk-Based Capital requirements, and they are, in the United States, generally different for banking institutions and different for insurance firms. Requirements for investment companies, such as mutual funds, are generally not needed, as their customers do not have guarantees of the kind enjoyed by the customers of banks and insurance firms. We will return to the process of calculation of Risk-Based Capital later in these notes.
Additionally, there are private companies playing a semi-regulatory role by providing credit rating of businesses and governments. They are called rating agencies. The two most known rating agencies are: Standard & Poor’s and Moody’s. Additionally, there are these rating agencies: Fitch Ratings, A.M. Best (only for insurance companies), and Weiss Research. Standard and Poor’s uses the following classification of credit risks:

- Investment grade: AAA, AA, A, BBB.
- Below investment grade (a.k.a., “junk”): BB, B, CCC, CC, C, and D (D = in default).

Moody’s uses the following classification of credit risks:

- Investment grade: Aaa, Aa, A, Baa.
- Below investment grade (a.k.a., “junk”): Ba, B, Caa, Ca, C, and D (D = in default).

12. Interest rate risk management
There are two main techniques of interest rate risk management:

- Cash flow matching. This is a technique designed to eliminate the financial intermediary’s risk. Under this approach, the scheduled negative cash flows produced by the liabilities are projected and then a portfolio of assets producing the same cash flows is purchased. Clearly, the assets at hand must be sufficient for such a purchase. In practice, a perfect and complete projection of cash flows is not always possible. Note that if even such a complete projection is possible, there is the problem of finding the least costly portfolio satisfying the conditions.
- Immunization. This technique calls for matching either nominal durations or durations of assets and liabilities.

We will now turn to practical issues related to implementation of immunization and related strategies, but before we do, let us recall some things we have learned by doing the mathematics:

- More dispersed sets of cash flows tend are more convex, and M-squared is actually the measure of dispersion,
- Longer duration cash flows tend to be more convex,
- Barbell (portfolio of one short-term zero coupon bond and one long-term zero coupon) has more convexity than bullet, if they have the same duration.
- An asset with deterministic positive cash flows (such asset must be option-free and default-free) must have positive convexity:

\[
C = \frac{1}{(1+i)^2} \sum_{t \geq 0} w_t (t+1).
\]

By the way, can an asset have negative duration? Yes, if its value declines when interest rates fall, and increases when rates rise. Interest Only (IO) strips, mortgage derivatives, do exhibit such behavior. Can an asset have negative convexity? Yes, if its duration declines when interest rates fall, and increases when rates rise. Duration is, in its first intuitive approximation, weighted average time to maturity (divided by 1+i). Generally, residential mortgages are prepaid if rates fall, and extend when rates rise. Thus negative convexity is quite natural to occur in portfolios of mortgages, mortgage-backed
securities, and other securities with similar prepayment options (bonds with sinking fund provision, and, although not to as great of an extent as for mortgages, callable bonds).

And … negative convexity lies at the very heart of life insurance business. Why? Because when you hold a portfolio of assets and liabilities, the duration of your portfolio is the weighted average duration of assets and liabilities, and the convexity of your portfolio is the weighted average convexity of assets and liabilities. If your assets are roughly equal to your liabilities in their market value, then if assets have longer duration than your liabilities, your net position has positive duration. If your assets have more convexity than your liabilities than your net position has positive convexity. But the converse is also true. Therein lies the problem: People live longer than bonds and mortgages. People-related liabilities, especially annuities in the payout phase, produce greater dispersion of cash flows than any bond or mortgage available. And that’s before you even mention any options embedded in either assets or liabilities.

But we should present the basic concepts of immunization, finally. Classical immunization begins with this idea: If \( f(x) \) is a function of a variable \( x \), and if the derivative \( f'(x) \) exists, then we have the following approximate identity: \( f(x + \Delta x) \approx f(x) + f'(x)\Delta x \). If the derivative equals zero, a small change in \( x \), denoted here by \( \Delta x \), results in almost no change in the value of the function. If we denote the market value of assets of the firm by \( A(i) \), where \( i \) denotes the effective annual rate of interest, and the market value of liabilities as \( L(i) \), then the surplus of the enterprise \( S(i) \) equals \( A(i) - L(i) \). If we apply the reasoning presented above to the function \( S(i) \), then we should manage assets and liabilities in such a way that \( A'(i) = L'(i) \). This would result in the enterprise being immunized from small changes in interest rates; in other words, the value of the surplus would not change given a small change in interest rates.

Suppose we want to minimize the change in the absolute level of surplus under a small change in interest rates. This means that we want \( \Delta S \approx 0 \) for \( \Delta i \approx 0 \). This does imply that \( S'(i) = A'(i) - L'(i) = 0 \). But \(-P'(i) = P(i) \cdot D\) for any security, this is the nominal duration of a security and represents a dollar change in the value of the security in response to a very small unit change in the interest rate. The strategy of immunizing the dollar value of the surplus calls for setting the dollar duration of the asset portfolio equal to the dollar duration of the liabilities portfolio.

But the regulatory constraints on the company’s surplus are typically expressed in terms of its capital ratio, not actual surplus level. The second approach to classical immunization calls for preserving a company’s capital ratio, or surplus ratio, that is, the ratio \( S(i)/A(i) \). Let \( k \) be the initial value of capital ratio of the firm. Note that immunization of the capital ratio \( k \) is equivalent to immunization of the ratio of liabilities to assets \( L(i)/A(i) = 1 - k \), or equivalently, the natural logarithm of that expression \( \ln(1 - k) \). Under an infinitesimal change of the interest rate, this implies that we want to set:

\[
\frac{d \ln(A(i))}{di} = \frac{d \ln(L(i))}{di}.
\]

Immunization summarized:
- To protect the absolute surplus level, set:
  \( A'(i) = L'(i) \),
  that is, dollar duration of assets equal to the dollar duration of liabilities, and
\[ \frac{d^2 A}{dt^2} > \frac{d^2 L}{dt^2} \]
that is, choose assets with more dollar convexity than the liabilities.

- To protect the surplus ratio level, set:
  \[ \frac{d \ln(A(i))}{dt} = \frac{d \ln(L(i))}{dt}, \]
  that is, duration of assets equal to the duration of liabilities, and,
  \[ \frac{d^2 \ln(A(i))}{dt^2} > \frac{d^2 \ln(L(i))}{dt^2}, \]
  that is, choose assets with more \( M^2 \) (dispersion) than that of liabilities.

Immunization, as defined above, has some problems. The standard list of such problems is as follows:
- The model assumes one interest rate, i.e., flat yield curve.
- The model assumes parallel shift of the yield curve, ignores other types of changes.
- The model assumes only instantaneous infinitely small change in the yield curve.
- Immunization requires continuous costly rebalancing.

But there is also a non-standard list, less commonly mentioned. This is what we hope for:
\[ S'(i_0) = 0, S''(i_0) > 0. \]

But in reality:
- People live longer than bonds and mortgages.
- Bonds and mortgages do not have enough convexity, and may take away convexity.
- If you buy convexity, you give up yield, if you sell convexity, you get extra yield.
  And … yield matters, especially for statutory accounting.
Assets are likely to have negative convexity at low interest rates because of:
- Prepayments of mortgages (in mortgages, MBS, mortgage derivatives).
- Callable bonds.
- Bonds with sinking fund provisions.
- Municipal bonds (for P/C companies) can also be callable.
By the way, you can buy convexity with puttable bonds, bond warrants, interest rate caps and floors, adjustable rate preferreds, and with very long term zero-coupon bonds. But you always give up yield!

Liabilities have more convexity not just because people live longer than bonds, but also because:
- As rates fall, life insurance liabilities have minimum interest rate guarantees.
- As rates rise, companies tend to follow the market in their credited rates (unless they want to experience severe disintermediation);
- Policyholders, especially in the case of deferred annuities, own a puttable bond.
The company may actually lose money no matter which way interest rates move.

13. Short straddle model
A position consisting of long put and long call is called a straddle. Its payoff looks like this:

\[ \text{Payoff} \]

\[ S - X \]
But … this ignores the premium paid for both options. With premium, the payoff looks like this:

The underlying must move in order for a straddle to make money. If the underlying does not move, the premium is wasted. But... roughly 2/3 of traded options expire worthless. Who makes money on them? Option writers, who are short the positions. Can you be short a straddle?

Life insurance companies are:
- Short prepayment/call options on interest rates on the asset side;
- Short the minimum interest rate guarantee (put) on the liabilities side;
- Short non-forfeiture benefit/guaranteed values put on the liabilities side;
- Short the option to earn market rates as the credited rate.

They do not show these options on their balance sheet, but are paid for the options in a way that produces income on their statutory income statement.

Is this a good deal or a bad deal? It all depends on how you look at it, after all this is a voluntary bargain in the free marketplace between two parties. But when combined with duration matching, this becomes dangerous. Note this: immunization seeks to remove risk, and earn profits under any change of interest rates, riskless profits, and if assets equal liabilities, those riskless profits are earned without capital outlay. This is, of course, standard definition of arbitrage. In practice, it ends up too often inverse arbitrage: “riskless” loss of surplus under any change of interest rates. Why? Negative convexity of surplus is quite dangerous if you match durations.
The problem is not just that when you match duration and you have the profile as above, you lose money when rates change. Because after rates fall, you will have much shorter duration of assets than that of liabilities. You will need new assets. If you again match durations, and again assume negative convexity of surplus (or you can give up yield and face rapid disintermediation, it’s your choice), and rates rise again, you will lose surplus again. And if rates just return to where they were when you started, you will have lost a portion of your surplus:

- Twice, and
- For good, because these “convexity losses” cannot ever be recovered.

The reason why “convexity losses” can never be recovered is very simple: the assets called or prepaid at low rates can never be bought back.

What we see is that immunization is actually pursuit of arbitrage, which in practice ends up being “inverse arbitrage”. In fact, as shown by Milgrom and Shiu, multivariate immunization is yet another pursuit of arbitrage. Does it lead to inverse arbitrage? Luckily, it just seems to lead to dedication. Of course, dedication is yet another pursuit of arbitrage. And, arbitrage is not a viable business model. We are not in arbitrage business. We are in derivatives business.

Is there any value then in calculating key rate durations as presented above? They do represent sensitivities of financial instruments with respect to certain changes in the yield curve, and, as such, they do convey some information. For example, the gradient vector formed from partial durations of the surplus does indicate the direction of change in interest rates to which such surplus is most sensitive (this follows from a well-known property of the gradient vector shown in elementary calculus).
The same is true about duration and convexity. Immunization may not be a viable business model, but knowing your duration and convexity helps you understand the risks you are facing. You may want to trade some of option-writing profits for a positive duration mismatch, as positive duration mismatch lets you earn profits from rolling down the yield curve ... most of the time. Or, if you know what you are doing, selling convexity (which is what Short Straddle is about) is fine, just don’t full yourself that it is riskless.

Concentrate on pricing and trading risks, instead of eliminating them. For example, non-American financial institutions are much more concerned with C-1 risk, and much less with C-3 risk. The reason why American mortgages, bank loans, and bonds have less credit risk is precisely because they have more prepayment risk. After all, prepayment under low rates is what protects the credit of the borrower. When the economy is in a recession... everyone gets a raise because of refinancing mortgages, and credit risk is reduced. The Risk-Based-Capital formula wisely combines C-1 and C-3 because they are “complementary goods”:

$$\sqrt{(C-2)^2 + ((C-1) + (C-3))^2 + (C-4)}.$$  

14. Shortfall Risk

Over the period 1926-1987, S&P 500 had a 6.8% return advantage over corporate bonds. But over the last 15 years of that period, stocks underperformed T-Bills in 35% of 6 to 18 month periods. The problem with investing in a risky asset, such as equities, is that a minimum return cannot be assured. This methodology, developed by Leibowitz, Bader and Kogelman, proposes controlling the probability of return falling below a specified level (called shortfall).

For simplicity, assume that you invest in equities and bonds only, in a one period model. Bonds are assumed risk-free for that one period. Assume that the risk free rate of return is $r_F$ and that the equity portion of the portfolio (the risky asset) has a normal distribution with mean $r_E$ and standard deviation $\sigma_E$. Then returns of a portfolio composed of portion $w$ placed in risky equities and $(1 - w)$ in risk-free asset will have a normal distribution with mean $r_p = (1 - w)r_F + wR_{E}$ and standard deviation $\sigma_p = w\sigma_E$. If we denote the random return of such portfolio by $R_p$ then we desire: $\Pr(R_p < \alpha) = 0.10$.

This is the essence of the shortfall methodology: find all portfolios for which the 10-th (or other prescribed) percentile of their probability distribution of returns is as needed, at an appropriate “significance level” $\alpha$.

We will derive the relationship between this standard deviation of the portfolio and the expected return for portfolios, which are at the 10th percentile of the probability distribution. We have $R_p = (1 - w)r_F + wR_{E}$, where $R_{E}$ is the random return of equities. Note the following:
\[
\Pr(R_p < \alpha) = \Pr((1-w)r_F + wR_E < \alpha) = \\
= \Pr\left(\frac{(1-w)\alpha + w\alpha - (1-w)r_F - w\sigma^2}{\sigma E} < \frac{\alpha - (1-w)r_F - wr_E}{\sigma E}\right) = \\
= \Pr\left(\frac{R_E - r_E}{\sigma E} < \frac{\alpha - (1-w)r_F - wr_E}{w\sigma E}\right) = 0.10.
\]

But we know that the 10th percentile of the standard normal distribution is \(-1.28\). So we have:
\[
\frac{\alpha - (1-w)r_F - wr_E}{w\sigma E} = -1.28,
\]
which implies that \(\alpha - (1-w)r_F - wr_E = -1.28w\sigma_E\) and consequently \(r_p = \alpha + 1.28\sigma_p\).

This is an equation of a line. Its y-intercept is the minimum return constraint \(\alpha\). It traces out all portfolios that have their 10th percentile of returns at the level \(\alpha\), and it shows it in the \((\sigma_p,r_p)\) coordinates.

**Exercise.**

Given the risk-free rate of \(r_F = 0.05\) and the equity expected rate of return of \(r_E = 0.09\), and equity standard deviation of \(\sigma_E = 0.20\), and the assumption of a flat yield curve, what is the composition of and the expected return of the portfolio that maximizes the expected return, but has only 10% probability of a value decline in a single year? Assume normal distribution of returns.

**Solution.**

The point is that the portfolio must lie on the efficient frontier which is composed of the risk-free asset and the risky equity, and that’s a straight line with intercept 0.05, and slope 0.04/0.20 = 0.20, i.e., the line \(r_p = 0.05 + 0.2\sigma_p\). Combination portfolio of equities and risk-free asset has the 10th percentile of its distribution equal to \(\alpha\) if \(E(R_p) = r_p = \alpha + 1.28\sigma_p\). We want \(\alpha = 0\), so that \(E(R_p) = r_p = 1.28\sigma_p\). Where these two lines intersect, expected return is maximized, and the 10th percentile is 0. Thus \(0.05 + 0.2\sigma_p = 1.28\sigma_p\), or \(0.05 = 1.08\sigma_p\), and therefore \(\sigma_p = 0.046296296\), which is about 23.15\% of 0.20 standard deviation of pure equity, indicating that this optimal portfolio is roughly 23.15\% equities and 76.85\% risk-free asset.
How would you use this methodology for an asset-liability portfolio? You should consider surplus as the portfolio of the financial intermediary. Then \( S = A - L \), and for the dollar returns we also have \( R_s(S) = R_s(A) - R_s(L) \) (if you let me invent this notation). But how do we define the rate of return of surplus? Leibowitz, Bader and Kogelman suggest:

\[
\text{Surplus Return} = \frac{S_1 - S_0}{L_0} = \frac{A_1 - A_0}{A_0} - \frac{L_1 - L_0}{L_0} = R_A \cdot FR_0 - R_L
\]

Here, \( FR \) is the initial funding ratio of the plan. Then if the liabilities have a random return, we need to find all portfolios whose 10\(^{th}\) percentile of surplus return is at the prescribed level (threshold). In their works, Leibowitz, Bader and Kogelman assume that all returns are normal and derive the following portfolios satisfying the surplus shortfall constraint \( \Pr(R_A \cdot FR_0 - R_L < \alpha) = 0.10 \).

In this model, all bonds, regardless of maturity, have the same expected return, and volatility of a bond portfolio is estimated as 1.5\% times duration.

Allowing lower threshold expands the “egg”, allowing for more efficient portfolios in asset-only context. Having higher funding ratio expands the “egg” and moves it to the left. Lower funding ratio forces closeness to the immunizing portfolio, but funding ratio under 100\% can’t be helped with immunization, or anything else.

15. Value at Risk
This methodology or risk management is based on the same principle as shortfall constraint: trying to control risk by studying the tail of the distribution. \textit{Value-at-Risk} is defined as the number VAR such that:

\[
\text{Probability (Portfolio loses more than VAR within time period } t) < \alpha,
\]
given:
- amount of time \( t \), and
- probability level \( \alpha \) (confidence level)
- all this under normal market conditions!

For example, if we know that:
Probability ($1 million in S&P 500 Index will decline by more than 20% within a year) < 10%

then Value at Risk (VaR) = $200,000 (20% of $1,000,000) with $\alpha = 0.10, t = 1$ year (typically time period is much shorter, expressed in days). VaR is typically a dollar amount, not %. Value at Risk is only about Market Risk under normal market conditions. VAR is important because it is used to allocate capital to market risk for banks, under their Risk Based Capital requirements. There are three basic VaR methodologies:
- Parametric;
- Historical;
- Simulation.

How is parametric VaR calculation performed?
- Estimate historical parameters: asset returns, variances and covariances, for all asset classes, or assets comprising the portfolio;
- Calculate portfolio expected return and standard deviation;
- Estimate VAR assuming normal distribution of portfolio return.

This procedure typically assumes normality and serial independence. It is wrong theoretically, but practitioners do not care about this, and it is quite simple. It may suffer from problems with estimating parameters, especially volatility.

How is historical VaR calculation performed?
- Assemble and maintain historical database;
- Use historical data as the future distribution.

The key issue: What if the future isn’t what it used to be? But … if generalize this procedure to the nonparametric method of bootstrap, this may be the best there is. Of course, bootstrap is rarely used in practice, because practitioners generally do not know what it is.

How is simulation VaR calculation performed?
- Specify distributions of model input factors,
- Use Monte Carlo simulation for factors,

Combine them into global outcome, get a probability distribution.

In this methodology, assumptions on factors and underlying probability distributions of those factors are crucial. If one can get that distribution ideally, this may be an ideal method.

We will now look at a practical VaR calculation using the parametric methodology, the most common approach. It relies heavily on volatility estimate, and assumes that consecutive daily returns are independent and identically distributed (IID). Because of this assumption, one can add volatilities of daily returns, and arrive at the following relationships: $\sigma_{yr} = \sigma_{day} \sqrt{252}$ and $\sigma_{day} = \frac{\sigma_{yr}}{\sqrt{252}}$. There are 252 trading days in a year, that’s why you see 252 in the formula, and clearly this approach relies on the assumption that volatility on non-trading days is minimal if nonexistent.
It is customary to assume in parametric VaR calculations that the expected return over period considered is 0% (because in practice calculations are done over very short periods). It is also customary to assume normal distribution of returns (this is, in fact, the essence of the parametric method).

Consider $10 million in stock A, N = 10 days (two trading weeks), and X = 99% confidence level (X = 1 - α, i.e., α = 1%). Assume daily volatility of 2%, i.e., daily standard deviation (SD) of $200,000. Under this model assumptions, 10 days standard deviation is \( \sqrt{10} \cdot 200,000 = 632,456 \). Because the 1st percentile of the standard normal distribution is -2.33 (and the 99th percentile is 2.33), VaR of this stock A portfolio is: 2.33 \cdot 632,456 = $1,473,621. Now consider a $5 million in stock B. Assume its daily volatility is 1%. Then its 10 day SD is $50,000 \sqrt{10}$. Its VaR is

\[
2.33 \cdot 50,000 \sqrt{10} \approx $368,405.
\]

Now combine the two assets in a portfolio and assume that the correlation of their returns is 0.7. We have (for 10-day standard deviation)

\[
\sigma_{A+B} = \sqrt{\sigma_A^2 + \sigma_B^2 + 2 \rho \sigma_A \sigma_B}
\]

and we get \( \sigma_{A+B} = $751,665 \). Thus 10-day 99% VaR of the combined A + B portfolio is

\[
2.33 \cdot 751,665 = $1,751,379.
\]

The amount

\[
($1,473,621 + $368,405) - $1,751,379 = $90,647.
\]

is the VaR benefit of diversification.

**Exercise.**

Consider a position consisting of $1,000,000 investment in asset X and $1,000,000 investment in asset Y. Assume that the daily volatilities of both assets are 0.1% and that the correlation coefficient between their returns is 0.30. What is the 5-day 95% Value at Risk for this portfolio?

**Solution.**

The standard deviation of the daily dollar change in the value of each asset is $1,000. The variance of the portfolio’s daily change is:

\[
1000^2 + 1000^2 + 2 \cdot 0.3 \cdot 1000 \cdot 1000 = 2,600,000.
\]

The standard deviation of the portfolio’s daily change in value is the square root of 2,600,000, i.e., $1,612.45. The standard deviation of the five-day change in the portfolio value is: $1,612.45 \cdot \sqrt{5} = $3,605.55. The 95th percentile of the standard normal distribution, from its table, is 1.645. Therefore (assuming zero mean), the five-day 95% Value at Risk is: 1.645 \cdot $3,605.55 = $5,931.

A linear model

Consider a portfolio of assets such that the changes in the values of those assets have a multivariate joint normal distribution. Let \( \Delta x_i \) be the change in value of asset \( i \) in one day, and then for the change in value of the portfolio \( \Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i \) is normally
distributed because of the multivariate joint normal distribution. Since $E(\Delta x_i) = 0$ is assumed for every $i$, $E(\Delta P) = 0$. We also have: $\sigma_i = \sqrt{\text{Var}(\Delta x_i)}$, $\rho_{ij} = \text{Corr}(\Delta x_i, \Delta x_j)$ and $\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j$. Therefore the 99% VAR for $N$ days is: $2.33 \sigma_p \sqrt{N}$.

VaR for portfolios of bonds

We can use duration to estimate volatility of bonds portfolios. Such duration estimation gives $\Delta P = -D \cdot P \cdot \Delta y$. Let $\sigma_y$ be the yield volatility per day. What is it? One way to look at it: SD of $\Delta y$. Then $\sigma_p = D \cdot P \cdot \sigma_y$. Another way of looking at it: SD of $\frac{\Delta y}{y}$, where $y$ is the zero coupon bond yield for maturity $D$. Then $\Delta P = -D \cdot P \cdot \frac{\Delta y}{y}$ so that $\sigma_p = D \cdot P \cdot \frac{\Delta y}{y} \sigma_y$. One can include convexity in this approach (in addition to duration), but this still does not account for nonparallel yield curve shifts.

An alternative to duration approximation is the procedure called cash flow mapping. It assumes that there exist certain standard maturity zero-coupon bonds for which we know their rates of return, volatilities and correlations (with each other) very well. Then any other bonds is “mapped” into a portfolio of such standard zero-coupon bonds as follows.

Consider a $1$ million Treasury maturing in 0.8 years, with 10% semi-annual coupon. It can be viewed as a 0.3 year $50K$ zero plus 0.8 year $1,050K$ zero. Suppose that the rates and zero prices are as follows:

<table>
<thead>
<tr>
<th></th>
<th>3 mos.</th>
<th>6 mos.</th>
<th>1 year</th>
</tr>
</thead>
<tbody>
<tr>
<td>zero yield</td>
<td>5.50%</td>
<td>6.00%</td>
<td>7.00%</td>
</tr>
<tr>
<td>zero price volatility (% per day)</td>
<td>0.06%</td>
<td>0.10%</td>
<td>0.20%</td>
</tr>
</tbody>
</table>

Assume the following daily returns correlations

<table>
<thead>
<tr>
<th></th>
<th>3 mos.</th>
<th>6 mos.</th>
<th>1 year</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 mos.</td>
<td>1.00</td>
<td>0.90</td>
<td>0.60</td>
</tr>
<tr>
<td>6 mos.</td>
<td>0.90</td>
<td>1.00</td>
<td>0.70</td>
</tr>
<tr>
<td>1 year</td>
<td>0.60</td>
<td>0.70</td>
<td>1.00</td>
</tr>
</tbody>
</table>
What is the rate for 0.80 years? We interpolate between 0.5 and 1.0 years and get yield of 6.60%. If you interpolate daily volatility, you get 0.16%. We then try to replicate volatility of the 0.8 year zero with 0.5 year zero and 1 year zero, by a position of $\alpha$ in the 6 mos. zero and $1 - \alpha$ in the 1 year zero. Matching variances we get the following equation:

$$0.0016^2 = 0.001^2 \alpha^2 + 0.002^2 (1 - \alpha)^2 + 2 \cdot 0.7 \cdot 0.001 \cdot 0.002 \cdot \alpha (1 - \alpha).$$

This is a quadratic equation which solves for $\alpha = 0.3203$. The 0.8 year zero is worth $1.050K / 1.066^{0.8} = $997,662. $\alpha$ is the portion of this amount, i.e., $0.3203($997,662) = $319,589 which is allocated to the six months zero, and the rest, i.e., $0.6797($997,662) = $678,073 is allocated to the one year zero.

Now we perform the same calculations for the 0.3 year $50K zero. The 0.25-year and 0.50-year rates are 5.50% and 6.00%, respectively. Linear interpolation gives the 0.30-year rate as 5.60%. The present value of $50,000 received at time 0.30 years is $50,000 / 1.056^{0.3} = $49,189.32. The volatility of 0.25-year and 0.50-year zero-coupon bonds are 0.06% and 0.10% per day, respectively. Using linear interpolation we get the volatility of a 0.30-year zero-coupon bond as 0.0068% per day. Assume that $\alpha$ is the value of the 0.30-year cash flow allocated to a 3-month zero-coupon bond, and $1 - \alpha$ is allocated to a six-month zero-coupon bond. We match variances obtaining the equation:

$$0.00068^2 = 0.0006^2 \alpha^2 + 0.001^2 (1 - \alpha)^2 + 2 \cdot 0.9 \cdot 0.0006 \cdot 0.001 \cdot \alpha (1 - \alpha),$$

which simplifies to $0.28\alpha^2 - 0.92\alpha + 0.5376 = 0$. This is a quadratic equation, and its solution is:

$$\alpha = \frac{-0.92 + \sqrt{0.92^2 - 4 \cdot 0.28 \cdot 0.5376}}{2 \cdot 0.28} = 0.760259.$$  

This means that a value of 0.760259($49,189.32) = $37,397 is allocated to the three-month bond and a value of 0.239741($49,189.32) = $11,793 is allocated to the six-month bond.

This way the entire bond is “mapped” into positions in standard maturity zero Treasury bonds. Since we are given the volatilities and correlations of those bonds, and since we are assuming zero return in a short period of time, we just take the portfolio as mapped, and calculate its standard deviation. The 10 day 99% VAR of the bond is then 2.33 times $\sqrt{10}$ times the SD calculated.

The linear model is only appropriate for situations when the portfolio returns are linearly related to certain market variables. Thus it may be inappropriate for certain derivatives, such as options, which have asymmetrical payoffs.

Stress testing and back testing

**Stress testing**: Estimating how the portfolio would have performed under the most extreme market conditions. For example: five standard deviation move in a market variable in a day. This is next to impossible under normal distribution assumption (happens once in 7000 years) but in reality it happens about once every 10 years (and some researchers say that one such move in some market happens every year).
Back testing: checking how well our model did in predicting things in the past. For example: how often did a one day loss exceed 1-day 99% VAR. If this happens roughly one percent of the time, we should be satisfied with our model.

16. Risk-Based Capital Requirements

Value-at-Risk calculation is performed by all banks because its result is generally equal to the level of capital required of banks by banks regulators. This approach of requiring a different level of capital depending on the level of risk chosen by the bank is a new regulatory framework adopted internationally in the 1990s. Similar approach has also been gradually introduced for insurance companies in the United States. However, the calculation of the required amount of the Risk-Based Capital (RBC) for insurance firms is dramatically different. We will outline it now.

The (United States) National Association of Insurance Commissioners (NAIC) instituted its RBC system for life insurance companies in 1993, followed by a property-casualty system in 1994 and a health system in 1998. The NAIC’s RBC system consists of two parts:

- A formula that is used to set a regulatory minimum capital level for each insurer, based on that insurer’s mix of assets, liabilities, and risk, and
- Definition of “financial impairment” and remedies to state insurance regulators in the event that an insurer meets that definition of impairment.

Formulas continuously evolve. NAIC publishes newsletters and guidelines for the calculation of Risk-Based Capital. The RBC system is meant to be a supplement, not a replacement, for the existing fixed minimum capital requirements that exist in each state. That is, the RBC formula requirements can be higher or lower than the fixed minimum capital requirements (which are typically $1 to $2 million), but each insurance company must meet both sets of standards. Many small insurance companies generate RBC requirements that are lower than the fixed dollar minimums, but for virtually all medium-sized and large insurers, the capital requirements generated by the RBC formula are higher than the state fixed minimums.

The RBC requirement (level of capital required in view of risk undertaken) is calculated by multiplying risk factors by values from the company’s financial statements, adding the results together, and then adjusting for covariance between major risk categories. The formula results are compared to the risk-adjusted capital of the insurer to develop the RBC ratio, which is the ratio of risk-adjusted capital to RBC. The ratio results are used to determine the degree to which an insurance company’s surplus is impaired. The model act specifies a series of increasingly stringent regulatory responses, as the RBC ratio decreases below 200%. A trend test is included to test whether insurers that were between the 200% breakpoint and 250% level were trending downward, which will trigger regulatory action, but an RBC ratio over 250% for a life company is sufficient to receive a passing grade on this pass/fail test.

There are four “action levels” under the NAIC RBC system.

- Company Action Level (CAL). If this level is reached, insurer is required to automatically submit a written, detailed business plan within 45 days that details the causes and actions that have led up to the capital impairment as well as a plan for the restructuring of the insurer’s business to rebuild capital to acceptable
levels. Alternatively, the company can detail plans to reduce its risk to a level commensurate with its actual capital level.

- Regulatory Action Level (RAL). In this case, insurer must conform to the requirements stated in the Company Action Level, and in addition is subject to an immediate regulatory audit. The regulator can then issue protective orders to force the insurer to either lower its risk profile or increase its capital to a level commensurate with its risk. A company that has reached the Company Action Level and that does not conform to the statutory requirements spelled out in the statute is also automatically deemed to have triggered the Regulatory Action Level.

- Authorized Control Level (ACL) is triggered by having statutory capital that is less than the Authorized Control Level RBC, as computed by the RBC formula or by failing to meet regulatory requirements imposed by the Regulatory Action Level. The Authorized Control Level is the capital level at which the state insurance commissioner is authorized, although not required, to place the insurance company under regulatory supervision.

- Mandatory Control Level. When that happens, the state regulator is required by statute to take steps to place the insurer under regulatory supervision.

Risk categories in the Life RBC formula

Originally, the major risk categories in the Life RBC formula were C1 – Asset Risk, C2 – Insurance Risk, C3 – Interest Rate Risk and C4 – Business Risk. These generic categories have been later refined and currently they are:

- C0 Affiliates Risk
- C1cs Asset Risk – Unaffiliated Common Stock
- C1o Asset Risk – Other Assets Risk
- C2 Insurance Risk
- C3a Interest Rate Risk
- C3b Health Credit Risk
- C4a General Business Risk
- C4b Administrative Expense Risk

The values calculated for each category are then combined in what is commonly called the covariance formula. The results of the covariance formula produce the Company Action Level RBC capital requirement. The Company Action Level requirement is twice the Authorized Control Level requirement.

If the insurer’s Total Adjusted Capital is less than the Authorized Control Level RBC requirement, the regulator is authorized to seize control of the company. The ACL RBC and the Total Adjusted Capital are both reported in the Five-Year History page of the annual statement. The RBC formula inputs and calculations are not made public.

Total Adjusted Capital = Statutory Capital & Surplus + Asset Valuation Reserve (AVR) including AVR in separate accounts + Half of company’s liability for dividends + company’s ownership
share of AVR of subsidiaries + Half of company’s ownership share of subsidiaries’ dividend liability

Separate risk-based capital models apply to life companies, property/casualty companies and health organizations. The common risks identified in the NAIC models for all types of companies include Asset Risk-Affiliates, Asset Risk-Other, Credit Risk, Underwriting Risk, and Business Risk.

Steps in RBC calculation are:
- Apply risk factors against annual statement values,
- Sum risk amounts and adjust for statistical independence (using the covariance formula),
- Calculate Authorized Control Level Risk-Based Capital amount,
- Compare ACL RBC to Adjusted Capital.

Total Adjusted Capital (Actual Capital) is divided by Authorized Control Level RBC (Hypothetical Minimum Capital) to get the RBC Ratio
- No Action (98% of companies) -- TAC/RBC over 200%
- Company Action Level -- TAC/RBC is 100% to 200%
- Authorized Control Level -- TAC/RBC is 70% to 100%
- Mandatory Control Level -- TAC/RBC is less than 70%

After the calculation of RBC, the company is also expected to perform sensitivity tests to indicate how sensitive the results are to certain risk factors’ changes.

Major categories in life RBC formula:
  - C0 – Subsidiary Insurers Risk
  - C1 – Asset Risk
  - C2 – Insurance Risk
  - C3 – Interest Rate Risk
  - C4 – Business Risk

Major categories in property/casualty RBC formula:
  - R0 – Subsidiary Insurers Risk
  - R1 – Fixed Income Asset Risk
  - R2 – Equity Asset Risk
  - R3 – Credit Risk
  - R4 – Insurance Risk – Reserve Development
  - R5 – Insurance Risk – Written Premiums

Major categories in health RBC formula:
  - H0 - Insurance Subsidiaries Risk
  - H1 –Asset Risk
  - H2 – Insurance Risk
  - H3 – Credit Risk
  - H4 – Business and Admin Expense Risk
Life asset risk accounted for in the RBC formula are:

- Defaults on Fixed Income Investments
- Changes in Market Value of Equity Investments
- Unrecoverability of Reinsurance Balances
- Company-Specific Experience (e.g., Mortgage Experience Adjustment)
- Over-Concentration in Specific Asset Investments
- Additional Risk from Affiliated Investments

But the following important risks are not accounted for:

- Market Value Adjustments
- Quality of Investments, although the following do adjustments do occur:
  - Bond Factors Differ By NAIC Rating Class
  - Mortgages In Default Have Higher Factors
  - Certain Types of Reinsurance Have Higher Factors
- Common Stock Diversification
- Interest Rate Risk
- Quality of Reinsurance
- Duration/Convexity Risk (not in the asset formula, and even in the C-3 part, not directly)

An interesting issues is the RBC calculation for affiliate companies:

- Parent RBC charge equal to its prorated share of affiliate’s RBC
  - 100% ownership = 100% rollup of RBC
  - 50% ownership = 50% rollup of RBC
- Treats affiliate as an extension of parent

Asset risk (C1) calculation example:

Note: Risk factors are developed by an NAIC Advisory Group. They are based on simulation testing for portfolios of bonds and are intended to account for default risk only. In what follows we use factors that may not be exactly the ones currently used.

<table>
<thead>
<tr>
<th>Asset Portfolio</th>
<th>Factor</th>
<th>RBC</th>
</tr>
</thead>
<tbody>
<tr>
<td>NAIC Class 1, U.S. Government</td>
<td>0.000</td>
<td>$0.00</td>
</tr>
<tr>
<td>NAIC Class 1, non U.S.-Government</td>
<td>0.003</td>
<td>$3.00</td>
</tr>
<tr>
<td>NAIC Class 2</td>
<td>0.010</td>
<td>$10.00</td>
</tr>
<tr>
<td>NAIC Class 3</td>
<td>0.020</td>
<td>$20.00</td>
</tr>
<tr>
<td>NAIC Class 4</td>
<td>0.045</td>
<td>$45.00</td>
</tr>
<tr>
<td>NAIC Class 5</td>
<td>0.100</td>
<td>$100.00</td>
</tr>
<tr>
<td>NAIC Class 6</td>
<td>0.300</td>
<td>$300.00</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>$478.00</td>
</tr>
</tbody>
</table>
There is then a bond size adjustment factor, i.e., the resulting RBC is multiplied by a number \( f_{SIZE} \), which is determined by the number of bonds in the portfolio. Note: there are no adjustments for portfolio size for stocks and mortgages.

For stocks, the C1 RBC is derived by multiplying the total value of all stocks by a factor provided by NAIC (30% for life companies, 15% for property/casualty companies, but note that factors do change over time).

Life mortgages formula is based on Asset Valuation Reserve (AVR)
- Mortgage Experience Adjustment (MEA)
- Separate Risk Factors By Mortgage Type
  - Farm
  - Residential
  - Commercial
  - Restructured
- Separate Risk Factors By Quality
  - In Good Standing
  - Overdue
  - In Foreclosure

Life real estate formula:
- Separate calculations for each property, then add up to get the total.
- Large number of separate RBC factors depending on type and quality of property, very complex.
- Questionable accuracy.

Accounting for insurance risk C2:
- Life formula uses tiered factors to adjust for size differences.
- Life insurance - Apply factors against net amount at risk.
- Health insurance - Apply factors against earned premiums.
- Flat factor is applied against health insurance reserves (e.g., not size-based).
- Credit allowed for premium stabilization reserves.

Example of tiered premium insurance risk RBC calculation:
A company has $10,000,000 in net amount at risk. A factor of 3.5% is applied to the first $5 million, and 2.0% for the next $5 million. Then the insurance risk RBC for this company is: 3.5% of $5,000,000 plus 2.0% of $5,000,000, for a total of $175,000 + $100,000 = $275,000.

The C3 risk RBC calculation uses the asset-liability model used for year-end Asset Adequacy Analysis cash flow testing, or a consistent model. You start by running the scenarios (12 or 50) produced from the interest-rate scenario generator. The statutory capital and surplus, \( S(t) \), should be captured for every scenario for each calendar year-end of the testing horizon. For each scenario, the C-3 measure is the most negative of the series of present values \( S(t) \), using 105 percent of the after-tax one-year Treasury rates for that scenario. Then one should rank the scenario-specific C-3 measures in descending
order, with scenario number’s measure being the positive capital amount needed to equal
the very worst present value measure. Taking the weighted average of a subset (currently,
scenarios ranked 17th through 5th are used with weights 0.02, 0.04, ..., 0.16, ..., 0.04,
0.02) of the scenario specific C-3 scores derives the final C-3 factor for the 50 scenario
set. For the 12 scenario set, the charge is calculated as the average of the C-3 scores
ranked 2 and 3, but cannot be less than half the worst scenario score. There are also cases
when single scenario testing is allowed.

Finally, the C4 RBC is calculated generally as a small percentage of premiums, in the
range of 2%.

The original life RBC covariance formula was: $\sqrt{(C1 + C3)^2 + C2^2 + C4}$.
It has been since then changed to: $C0 + \sqrt{(C1 + C3)^2 + C2^2 + C4}$.

The property/casualty covariance formula is: $R0 + \sqrt{(R1 + R2 + R3 + R4 + R5)^2}$.
The result of this formula is used in the RBC ratio to compare to the Company Action
Level RBC (200% of Authorized Control Level). If you would rather compare to the
Authorized Control Level, the life covariance formula is:

$$\frac{1}{2} \left( C0 + \sqrt{(C1 + C3)^2 + C2^2 + C4} \right).$$

Historical results, life companies

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>No Action</td>
<td>1,532</td>
<td>1,515</td>
<td>1,494</td>
<td>1,440</td>
<td>1,210</td>
</tr>
<tr>
<td>Company Action Level</td>
<td>9</td>
<td>16</td>
<td>12</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>Regulatory Action Level</td>
<td>8</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Authorized Control Level</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Mandatory Control Level</td>
<td>9</td>
<td>7</td>
<td>3</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>Total</td>
<td>1,559</td>
<td>1,543</td>
<td>1,513</td>
<td>1,472</td>
<td>1,231</td>
</tr>
</tbody>
</table>

Historical results, property/casualty companies

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>No Action</td>
<td>2,348</td>
<td>2,346</td>
<td>2,359</td>
<td>2,353</td>
</tr>
<tr>
<td>Company Action Level</td>
<td>20</td>
<td>33</td>
<td>39</td>
<td>33</td>
</tr>
<tr>
<td>Regulatory Action Level</td>
<td>13</td>
<td>15</td>
<td>19</td>
<td>12</td>
</tr>
<tr>
<td>Authorized Control Level</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>Mandatory Control Level</td>
<td>27</td>
<td>22</td>
<td>25</td>
<td>16</td>
</tr>
<tr>
<td>Total</td>
<td>2,413</td>
<td>2,422</td>
<td>2,453</td>
<td>2,419</td>
</tr>
</tbody>
</table>

Source: Dr. Michael Barth, Georgia Southern University.
The Canadian equivalent of the U.S. RBC is called the Minimum Continuing Capital and Surplus Requirement (MCCSR). Its calculation is set by guidance provided by the Office of Supervision of Financial Institutions (OSFI). 120% of MCCSR is the acceptable level of capital for an insurance firm. But if a company wants to have higher ratings, must have higher MCCSR. The key components of MCCSR calculation are:

- Asset risk,
- Mortality-morbidity risk,
- Repricing risk,
- Interest rate risk.

As in Poland, Canadian companies are more concerned with credit risk than with interest rate risk, and credit risk usually contributes more than half of MCCSR. The calculation is done by applying factors to the book value of different asset classes according to their riskiness. As in the U.S., all federal government bonds have a 0% factor. But unlike the U.S., also provincial bonds have 0% factor.

Some issues in the calculation:
- There is no credit for diversification of assets,
- There is no differentiation among bonds once placed in a specific quality group (regardless of their maturity, or small differences in quality),

Effectively, assets should be evaluated based on their returns after consideration for the MCCSR factor. Return (or spread) alone is not enough. MCCSR position may be improved by securitizing assets. Also, MCCSR formula does not consider derivatives, and one may use derivatives to improve MCCSR position.

**Exercise**

**May 2003 SOA Course 6 Examination, Problem No. C-13**

You are given the following for an insurance company that currently offers term insurance and fixed deferred annuities:

- Corporate pre-tax target return on capital of 18%
- Risk-based capital (RBC) formula:

  \[ 1.5 \cdot \sqrt{ (\text{Asset Default Risk Component}^2 + \text{Mortality Risk Component}^2) } \]

**Asset Default Risk Component (C-1)**

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Amount (millions)</th>
<th>RBC Factor</th>
<th>Historical Mean Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond</td>
<td>600</td>
<td>1%</td>
<td>7%</td>
</tr>
<tr>
<td>Real Estate</td>
<td>300</td>
<td>7%</td>
<td>8%</td>
</tr>
<tr>
<td>Common Stock</td>
<td>100</td>
<td>20%</td>
<td>10%</td>
</tr>
</tbody>
</table>

**Mortality Risk Component (C-2)**

<table>
<thead>
<tr>
<th>Net Amount at Risk</th>
<th>Amount (millions)</th>
<th>RBC Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>10,000</td>
<td>0.1%</td>
<td></td>
</tr>
</tbody>
</table>

The industry-wide ratio of C-1 to C-2 is 1.5. The risk-free rate is 6%.

(a) Describe the shortcomings of this RBC formula.
(b) Calculate the RBC-adjusted spread for this company’s asset portfolio.
(c) Evaluate the competitive advantage of the company’s product lines from a cost of capital perspective.

Solution.
(a) Shortcomings:
- Limited asset classes,
- No component for asset-liability risk,
- No covariance of risks component,
- Flat reference to amount at risk,
- No reference to asset diversification within asset class,
- No reference to asset credit quality within asset class,
- Equal weights for asset and liability risks, probably not appropriate,
- No accounting for subsidiaries.
- Does not account for general business risks,
- This is just a formula, does not reflect the actual quality of company’s performance, such as, e.g., underwriting standards.

(b) What is the RBC ratio of this company?
We have:
\[ C_1 = 600 \cdot 1.0\% + 300 \cdot 7\% + 100 \cdot 20\% = 47, \]
\[ C_2 = 10,000 \cdot 0.1\% = 10. \]
Therefore the RBC required is:
\[ 1.5 \cdot \sqrt{47^2 + 10^2} \approx 72.08. \]
There is no information about the actual surplus held, so we will assume a 100% RBC ratio. Now consider the RBC formula in this problem:
\[ RBC = F(x, y) = 1.5 \sqrt{x^2 + y^2}, \]
where \( x \) is the \( C_1 \) value and \( y \) is the \( C_2 \) value. Suppose the \( C_1 \) value increases by \( \Delta x \). Then the new RBC is:
\[ RBC = F(x + \Delta x, y) = 1.5 \sqrt{(x + \Delta x)^2 + y^2}. \]
The resulting change in RBC is:
\[ 1.5\sqrt{(x + \Delta x)^2 + y^2} - 1.5\sqrt{x^2 + y^2} \approx \frac{\partial F(x, y)}{\partial x} \cdot \Delta x. \]
In the case of this RBC formula, we have \( \frac{\partial F(x, y)}{\partial x} = \frac{1.5x}{\sqrt{x^2 + y^2}}. \)
Let us assume that a company buys one unit of bonds. What is the amount of capital required for that? The \( C_1 \) amount would be increased by \( \Delta x = 0.01 \), as the RBC requirement for bonds is 1%. The increase in required capital is approximately
\[ \frac{1.5 \cdot 47 \cdot 0.01}{\sqrt{47^2 + 10^2}} = 0.01467. \] Hence one additional unit of bonds will earn the following spread over the risk-free rate, after adjusting for the cost of capital required, of:
Now assume that the company buys one additional unit of real estate. Then $\Delta x = 0.07$, as the RBC requirement for real estate is 7%. The increase in required capital is approximately

$$\frac{1.5 \cdot 47 \cdot 0.07}{\sqrt{47^2 + 10^2}} \approx 0.1027.$$ 

This one additional unit of real estate will earn the following spread over the risk-free rate, after adjusting for the cost of capital required, of:

$$(8\% - 6\%) \cdot 1 - (18\% - 6\%) \cdot 0.1027 = 0.7676\%.$$ 

Finally, assume that the company buys one additional unit of stocks. Then $\Delta x = 0.2$, as the RBC requirement for stocks is 20%. The increase in required capital is approximately

$$\frac{1.5 \cdot 47 \cdot 0.20}{\sqrt{47^2 + 10^2}} \approx 0.2934.$$ 

This one additional unit of real estate will earn the following spread over the risk-free rate, after adjusting for the cost of capital required, of:

$$(10\% - 6\%) \cdot 1 - (18\% - 6\%) \cdot 0.2934 = 0.4702\%.$$ 

The total spread is weighted by asset allocation:

$$0.60 \cdot 0.824\% + 0.30 \cdot 0.7676\% + 0.10 \cdot 0.4702\% =$$

$$= 0.4944\% + 0.2303\% + 0.0470\% = 0.7717\%.$$ 

This is about 77 basis points.

(c) A company enjoys a competitive advantage if it has a lower cost of capital. The capital required by the market is considered a sum of the “Face Capital”, which earns the risk-free rate, and “At-Risk Capital”, which earns the rate of return on stocks. The cost of capital for all of capital is the appropriate weighted average of the two. Because of the RBC formula:

$$1.5 \cdot \sqrt{\left(\text{Asset Default Risk Component}^2 + \text{Mortality Risk Component}^2\right)},$$

it makes sense for companies to equally balance the asset default risk component and the mortality risk component. Hence companies with relatively high proportion of $C_1/RBC$ would have advantage to sell more pure insurance risk type products like term insurance. Those with relatively low ratio have competitive advantage to sell accumulation products, as those are investment oriented, with relatively lower amount at risk. We should note that RBC requirement has the potential to tie down some capital and restrain company’s ability to grow and sell certain more capital-intensive products. This is particularly true if the RBC ratio is near the regulatory or rating agencies threshold level (e.g. trend test zone). In the case of this company $\frac{C_1}{C_2} = \frac{47}{10} = 4.7$, while for entire industry, $\frac{C_1}{C_2} = 1.5$, hence this company is now overly concentrated in $C_1$ risk, and it will be more competitive to sell purely insurance-oriented products.
Solvency II
Solvency II is the European Commission project which (quoting from the European Commission’s “Framework for Consultation”) aims to develop “a new solvency system to be applied to life assurance, non-life insurance and reinsurance undertakings, which Member States and supervised institutions are able to apply in a robust, consistent and harmonized way.”

A three-pillar system is proposed, similar to Basel II:
• Pillar 1: quantification of capital requirements;
• Pillar 2: supervisory review process; and
• Pillar 3: market analysis of published data.

Pillar 1 encompasses two capital requirements (MCR and SCR) sitting on top of technical provisions made up of the best estimate of the liability plus a risk margin, as shown in the following diagram taken from CEIOPS Consultation Paper 20:

There is an overview of the effect of Solvency II on France at this web site:
Original Solvency II timeline

This has been adjusted following the Credit Crisis of 2008, currently Solvency II is scheduled to come into effect early 2013.

Situation in the United States

The expression “technical provisions” is not used in American insurance terminology, instead these quantities are called reserves.

In the United States, until 1991, reserves were calculated based on assumptions prescribed by law (statutory assumptions). This has been changing since then, first with introduction of cash flow testing in 1991, and then Principles Based Reserving (PBR) in 2008.

Principles Based Reserving
The proposed framework in the United States would define the minimum reserve as the greater of the amounts calculated using a seriatim deterministic method (deterministic reserve) and a stochastic method when the underlying risks of the polices require a stochastic approach (stochastic reserve). Both the deterministic reserve and the stochastic...
reserve would be determined by taking the present value of net cash flows arising from the contract, where the net cash flows reflect all cash outflows (e.g. benefits, expenses, but excluding Federal Income Taxes) less all cash inflows (gross premiums and other revenue items).

**How do we manage risk of insolvency for a long-term insurance enterprise?**
Traditional approach: Conservative assumptions, providing sufficient margin to minimize the probability of insolvency.
Next step: Sensitivity analysis and management of sensitivity.
Ultimate: Fair value.

Examples:
• Conservative approach: low valuation interest rate, high mortality rate for life insurance, low mortality rate for annuities. Gradually abandoned, because while it may work for internal risk management, it deviates from market value of liabilities so much that with open access to the market instruments created arbitrage opportunities.
• Sensitivity analysis and management of sensitivity: New York Regulation 126.
• Fair value: Market value (reinsurance), or stochastic scenarios analysis, probability distribution of key variables (ultimate surplus, reserve).

**Solvency II capital requirement approach**

**Pillar 1 of the Solvency II** framework will set out two levels of capital requirements.

• The first of these is the Solvency Capital Requirement. This is a risk-based assessment of the level of capital that the business needs to operate. Should the capital fall below this level then the supervisor may intervene. The SCR is based on an assessment of insurance, credit, market and operational risk elements. The capital requirement is based on a 1 in 200 (99.5%) confidence level over a one-year time horizon. The capital requirement can be calculated using internal models (full or partial models are being considered) or a standard approach.

When the SCR has been calculated, it is subject to supervisory review and an adjusted SCR will be issued by the regulator (i.e., reflecting their view of how well the SCR reflects the risks in the business).

• The second capital requirement is the Minimum Capital Requirement. The exact role and nature of this is still under consultation. However, in simple terms it is the absolute minimum capital requirement, below which the regulator would ask the firm to cease writing business. The MCR will be based on a much simplified calculation basis, which is currently subject to consultation. It may be a simplified version of the SCR calculation or it may simply be a percentage of last year's SCR calculation.
Pillar 2 of Solvency II sets out the internal risk and capital management standards that firms should follow.

All firms will be expected to undertake an Individual Risk and Capital Assessment (IRCA) to identify the level of capital that the business requires. The use of an internal model is not expected if the standard model is being used under Pillar 1, but it is a quantitative as well as qualitative exercise.

The second aspect of this Pillar is the supervisory review process. Solvency II allows the supervisors to undertake an assessment of the adequacy of the assessment of the risk-based capital requirements and risk management processes and to give an adjusted SCR requirement to the firm.

Given that Pillar 2 gives the supervisors of the nation states review powers and the ability to adjust the capital requirements of a firm, it is a significant element of the framework. United Kingdom firms have experienced a similar regime but many European states have no equivalent regime. There are concerns that, notwithstanding the supervisory peer review process proposed, there may be inconsistencies in the application of the rules in the early stages of implementation. Given the scarcity of resources with experience of reviewing internal capital models this is not surprising.

Pillar 3 of Solvency II concerns the disclosures that firms will be expected to make. These take two forms – disclosures to the supervisors and public disclosures.

However, broadly the disclosures are expected to be in respect of the following areas:
- Business overview and performance,
- Governance,
- Valuation basis used for solvency purposes,
- Risk and capital management.

One of the key topics under consultation is the disclosure around the SCR. Under discussion is whether the adjusted SCR (i.e. after supervisory review) should be disclosed and whether the disclosure should split the original estimate and the supervisory loading. The requirements will undoubtedly mean disclosure of more information than is currently disclosed and will increase the reporting burden on firms significantly. A key issue under debate currently is how far the reporting intrudes into sensitive commercial areas and what elements of information should be available only to the regulator or should be publicly showed. There are some similarities with the current IFRS reporting proposals, but the Solvency II regime goes further in its requirements.

Quantitative Impact Studies
http://www.ceiops.org/content/view/118/124/

The European Commission has requested that CEIOPS investigate the quantitative impact of key changes to be implemented under the new Solvency II regime. A variety
of insurers and reinsurers were invited to take part in these tests and to feed back their results. The findings and details of the tests are summarized below.

QIS 1
http://www.solvency-2.com/KeyPoints/QIS1Home.php
Conducted at the end of 2005 to acquire insight into possible effects of the new solvency regulations, specifically pillar 1 issues, QIS1 focuses on the level of prudence in current technical provisions and provides feedback from insurers and reinsurers on the feasibility of initial proposals for the standard model.

QIS2
http://www.solvency-2.com/KeyPoints/QIS2Home.php
Building on the findings of QIS1, QIS2 investigates the effect on insurance undertakings of the possible restatement of the value of both assets and liabilities under the Solvency II framework, as well as some possible options for setting the capital requirement (MCR and SCR).

QIS 3
http://www.solvency-2.com/KeyPoints/QIS3_Home.php
Building on the previous impact studies QIS3 took a detailed look at group and calibration issues. There was particular focus regarding the appropriate choice of calibrations and factors along with the implications solvency II will have on company wide strategic structural decisions.

QIS 4
http://www.solvency-2.com/KeyPoints/QIS4Home.php
CEIOPS presented the results in November 2008. This study focused on the use of full & partial internal models as well as further group issues to help CEIOPS properly measure the impact of the future regime on the industry.

QIS5
Set for August 2010

Key points
• Solvency II is based on principles not rules.
• The firm’s governance and risk management must match its risk profile.
• An effective risk management system is key to governance.
• Internal models used for Solvency II must be embedded into the firm, including strategic decision-making.
• A risk management function, if not already in existence, needs to be established.

The SCR is intended to reflect all quantifiable risks that the firm might face, including:
• non-life underwriting risk

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-78-
- life underwriting risk
- special health underwriting risk
- market risk
- credit risk
- operational risk

It will correspond to the value-at-risk of the net assets (i.e., capital, owners’ equity) of the firm subject to a confidence level of 99.5% over a one-year period, assuming continued solvency.

Firms can calculate the SCR in one of two ways:
- use of the standard formula
- use of an internal or partial internal model

The standard formula has a module for each of the above risk types. An SCR for each risk-type (except operational risk) is calculated, with each module calibrated to the “one-year 99.5%” level. The results are then aggregated, with diversification effects governed by a correlation matrix. A separate loading for operational risk (details to be decided) is then added.

**Risk-Based Capital Requirements Worldwide**

There is a general world-wide movement towards setting capital requirements for financial intermediaries in terms of risks undertaken by them. In addition to Solvency II, other such developments are:
- Basel II accord for banks,
- Risk-Based Capital requirements for U.S. insurance companies (using so called *modular approach*, as opposed to company-internal models proposed in Solvency II).

**Base II** is the second of the Basel Accords, which are recommendations on banking laws and regulations issued by the Basel Committee on Banking Supervision. The purpose of Basel II, which was initially published in June 2004, is to create an international standard that banking regulators can use when creating regulations about how much capital banks need to put aside to guard against the types of financial and operational risks banks face. Advocates of Basel II believe that such an international standard can help protect the international financial system from the types of problems that might arise should a major bank or a series of banks collapse. In practice, Basel II attempts to accomplish this by setting up rigorous risk and capital management requirements designed to ensure that a bank holds capital reserves appropriate to the risk the bank exposes itself to through its lending and investment practices. Generally speaking, these rules mean that the greater risk to which the bank is exposed, the greater the amount of capital the bank needs to hold to safeguard its solvency and overall economic stability.

Objectives:
- Ensuring that capital allocation is more risk sensitive;
- Separating operational risk from credit risk, and quantifying both;
Attempting to align economic and regulatory capital more closely to reduce the scope for regulatory arbitrage.

While the final accord has largely addressed the regulatory arbitrage issue, there are still areas where regulatory capital requirements will diverge from the economic. Basel II has largely left unchanged the question of how to actually define bank capital, which diverges from accounting equity in important respects. The Basel I definition, as modified up to the present, remains in place.

The Accord in operation: Basel II uses three pillars
- (1) minimum capital requirements (addressing risk),
- (2) supervisory review and
- (3) market discipline – to promote greater stability in the financial system.

The Basel I accord dealt with only parts of each of these pillars. For example: with respect to the first Basel II pillar, only one risk, credit risk, was dealt with in a simple manner while market risk was an afterthought; operational risk was not dealt with at all.

The first pillar
The first pillar deals with maintenance of regulatory capital calculated for three major components of risk that a bank faces: credit risk, operational risk and market risk. Other risks are not considered fully quantifiable at this stage.
- The credit risk component can be calculated in three different ways of varying degree of sophistication, namely standardized approach, Foundation IRB and Advanced IRB. IRB stands for "Internal Rating-Based Approach".
- For operational risk, there are three different approaches - basic indicator approach or BIA, standardized approach or STA, and advanced measurement approach or AMA.
- For market risk the preferred approach is VaR (value at risk).

The second pillar
The second pillar deals with the regulatory response to the first pillar, giving regulators improved tools over those available to them under Basel I. It also provides a framework for dealing with all the other risks a bank may face, such as systemic risk, pension risk, concentration risk, strategic risk, reputation risk, liquidity risk and legal risk, which the accord combines under the title of residual risk.

The third pillar
The third pillar greatly increases the disclosures that the bank must make. This is designed to allow the market to have a better picture of the overall risk position of the bank and to allow the counterparties of the bank to price and deal appropriately.