Cohomology of Bloch-Kato profinite groups

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Outline

- Bloch-Kato profinite groups
- Main theorem: Cohomology of Bloch-Kato profinite groups
- Motivation: The Bloch-Kato conjecture
- Applications: Detecting absolute Galois groups
A pro-finite group $G$ is an inverse limit of finite groups $(G_\alpha)$

$$G = \lim_{\alpha} G_\alpha$$

**Theorem**

The following statements are equivalent for a group $G$.

1. **Algebraic**: $G$ is profinite
2. **Topological**: $G$ is compact Hausdorff and totally disconnected topological group.
3. **Arithmetical**: $G = \text{Gal}(L/K)$ of some Galois extension $L/K$. 
1. Bloch-Kato profinite groups

$G$ profinite group

$H^*(G, \mathbb{F}_p)$ continuous cohomology of $G$. (This is a graded commutative ring.)

$G$ is said to be a Bloch-Kato profinite group if $H^*(G, \mathbb{F}_p)$ is a quadratic algebra.

*Loosely speaking:* A graded algebra is **quadratic** if it is generated by degree 1 elements and its relations are generated by degree 2 elements.
Let $A_*$ be a graded commutative connected $\mathbb{F}_p$-algebra.

Consider the natural map

$$T^*(A_1) \rightarrow A_*.$$ 

$\bigoplus_r T_r$ belongs to the kernel, where $T_r$ is the submodule of $A_1^\otimes r$ generated by $a_1 \otimes \cdots \otimes a_r$ where $a_i a_j = 0$ in $A_2$ for some $i \neq j$.

This gives an induced map

$$\omega_A: T^*(A_1)/\bigoplus_r T_r \longrightarrow A_*.$$ 

$A_*$ is **quadratic** if $\omega_A$ is an isomorphism of $\mathbb{F}_p$-algebras.

**Note:** $A_*$ has generators in degree 1 and relations in degree 2.
$G$ be a profinite group and $p$ be a fixed prime.

The $p$-descending central sequence is:

\[ G = G^{(1)} \supset G^{(2)} \supset G^{(3)} \supset \cdots \supset G^{(n)} \supset \cdots \]

\[ G^{(n)} = \left[ G, G^{(n-1)} \right] \left( G^{(n-1)} \right)^p \]

\[ G \longrightarrow G/G^{(n)} (= G^{[n]}) \]

**Inflation maps:**

\[ H^*(G^{[n]}, \mathbb{F}_p) \longrightarrow H^*(G, \mathbb{F}_p) \]
2. The main theorem

Let $G$ be a Bloch-Kato profinite group. Then,

$$\inf : H^*(G^3, \mathbb{F}_p)_{dec} \cong H^*(G, \mathbb{F}_p).$$

The decomposable part $A_{dec}$ of a graded connected commutative algebra $A$ is the subalgebra of $A$ generated by degree 1 elements.

**Slogan:** $G^3$ determines $H^*(G, \mathbb{F}_p)!$
3. Motivation

$F$ – field that contains a primitive $p$–th root of unity.

$F_{\text{sep}}$ – the separable closure of $F$.

The ultimate goal of mankind: understand the structure of the absolute Galois group

$$G_F := \text{Gal}(F_{\text{sep}}/F).$$

$$G_\mathbb{C} = 1, \quad G_\mathbb{R} = C_2, \quad G_\mathbb{Q} = ???$$

Galois cohomology: $H^*(G_F, \mathbb{F}_p)$ is the continuous cohomology of the profinite group $G_F$.

Want a presentation of this rather mysterious Galois cohomology ring $H^*(G_F, \mathbb{F}_p)$ by generators and relations.
The Bloch-Kato conjecture

\( F^* \): multiplicative group of \( F \).

Milnor \( K \)-theory \( K_*(F) \): graded ring defined (1970) as

\[
K_*(F) := T(F^*)/\langle a \otimes b | a + b = 1 \rangle.
\]

Bloch–Kato conjecture: \( K_*(F)/p \cong H^*(G_F, \mathbb{F}_p)!! \)

\( H^*(G_F, \mathbb{F}_p) \) is generated by one-dimensional classes and the relations in the ring are generated by two-dimensional classes.

“Absolute Galois groups are Bloch-Kato profinite groups.”

- \( p = 2 \) (Milnor conjecture): proved by Voevodsky in 1996.
- \( p \) odd (Bloch-Kato conjecture): recently proved by Rost, Voevodsky, and Weibel.
4. Applications: Detecting absolute Galois groups

The Central burning question: When is a profinite group $G$ an absolute Galois group?

Our main theorem allows us to identify some profinite groups which cannot be the absolute Galois groups of any fields.

**Theorem**

*Let $F$ be a field which has a primitive $p$th root of unity. Then,*

$$H^\ast(G_F^\{3\}, \mathbb{F}_p)_{dec} \cong H^\ast(G_F, \mathbb{F}_p).$$

**Slogan:** $G_F^\{3\}$ determines Galois cohomology!

**Comment:** Proof of the theorem uses the Bloch-Kato conjecture.

Let $S$ be a free pro-$p$-group and let $R$ be a non-trivial closed normal subgroup of $S$ that is contained in $S^{(3)}$. Then $G = S/R$ cannot occur as an absolute Galois group.

Summary of the proof:

$$H^2(S, \mathbb{F}_p) = 0 \text{ and } H^2(G, \mathbb{F}_p) \neq 0$$

$$S^{[3]} \cong G^{[3]}$$

By our main theorem, at most one of the groups ($S$ and $G$) can be an absolute Galois group.

It is well known that $S$ is an absolute Galois group over a field $F$ of characteristic 0.

Therefore, $G$ is not realisable as an absolute Galois group.
Second family of non-relaisable groups


Let $G$ be a pro-$p$ group such that $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) < cd(G)$. Then $G$ is not an absolute Galois group of any field.

Proof.

Let $d = \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ (p odd).

$H^1(G, \mathbb{F}_p) \cong H^1(G^{[3]}, \mathbb{F}_p)$, $\therefore d = \dim_{\mathbb{F}_p} H^1(G^{[3]}, \mathbb{F}_p)$.

$H^{d+1}(G^{[3]}, \mathbb{F}_p)_{dec} = 0$ (cup product is graded-commutative)

On the other hand, $H^{d+1}(G, \mathbb{F}_p) \neq 0$.

Thus $H^*(G^{[3]}, \mathbb{F}_p)_{dec} \ncong H^*(G, \mathbb{F}_p)$.

By our main theorem $G$ is not an absolute Galois group. \qed
Thank You