Towards a refinement of the Bloch–Kato conjecture

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Outline

- Motivation
- The Bloch–Kato conjecture
- A refinement of the Bloch–Kato conjecture
- Applications
Let $F$ be a field that contains a primitive $p$–th root of unity and let $F_{\text{sep}}$ denote the separable closure of $F$.

The ultimate goal of mankind is to understand the structure of the absolute Galois group

$$\hat{G}_F := \text{Gal}(F_{\text{sep}}/F).$$

Cohomology is a powerful tools to study groups. So it is natural to investigate the structure of the (continuous) Galois cohomology $H^*(\hat{G}_F, \mathbb{F}_p)$ of the profinite group $\hat{G}_F$.

The Bloch–Kato conjecture gives a beautiful presentation of the rather mysterious Galois cohomology ring $H^*(F, \mathbb{F}_p)$ by generators and relations.
Let $F^*$ denote the multiplicative group of $F$. The Milnor $K$–theory $K^M_*(F)$ is a graded ring defined (1970) as

$$K^M_*(F) := T(F^*)/\langle a \otimes b \mid a + b = 1 \rangle.$$ 

The Bloch–Kato conjecture claims that $K^M_*(F)/p \cong H^*(F, \mathbb{F}_p)$!!

(When $p = 2$ this is called the Milnor conjecture — proved by Voevodsky in 2002.)

In particular, this tells us that the ring $H^*(F, \mathbb{F}_p)$ is generated by one–dimensional classes and the relations in the ring are generated by two–dimensional classes.
Consider the Kummer sequence of $\hat{G}_F$–modules:

$$1 \longrightarrow \mu_p \longrightarrow F^*_{\text{sep}} \xrightarrow{\chi^p} F^*_{\text{sep}} \longrightarrow 1.$$ 

The boundary map in the long exact sequence in Galois cohomology is a homomorphism $F^* \longrightarrow H^1(F, \mathbb{F}_p)$.

Bass and Tate proved (1973) that the Steinberg relations: 

$$(a) \cup (1 - a) = 0$$

for all $a$ in $F^* - \{1\}$ hold in $H^2(F, \mathbb{F}_p)$.

The map $F^* \rightarrow H^1(F, \mathbb{F}_p)$ therefore extends to a natural map $K_*(F) \rightarrow H^*(F, \mathbb{F}_p)$ which goes through $K_*^M(F)/p \rightarrow H^*(F, \mathbb{F}_p)$.

The Bloch-Kato conjecture: $K_*^M(F)/p \xrightarrow{\sim} H^*(F, \mathbb{F}_p)$. 

The norm residue homomorphism
The **Bloch–Kato conjecture** is a huge industry with a long, rich, interesting and convoluted history of over 40 years spreading in areas of arithmetic, geometry, number theory, and homotopy theory.

The Bloch–Kato conjecture is now a theorem of Voevodsky and Rost (some details necessary were given by Weibel).

An incomplete list of some other great mathematicians who contributed to the literature on the Bloch–Kato conjecture:

*Milnor, Bass, Tate, Bloch, Kato, Lichtenbaum, Beilinson, Suslin, Merkurjev, Izhboldin, Quillen, Swan, Levine, Morel, Orlov, Vishik, Friedlander, Arason, Jacob, Elman, Lam ......*
Let $F(p)$ denote the maximal $p$–extension of $F$.

The Bloch–Kato conjecture implies that

$$\inf: H^*(G_F, \mathbb{F}_p) \xrightarrow{\sim} H^*(\hat{G}_F, \mathbb{F}_p).$$

Therefore it is enough to study $H^*(G_F, \mathbb{F}_p)$. 
Consider the tower of fields

\[ F(p) \]
\[ \vdots \]
\[ F^{(n+1)} \]
\[ \vdots \]
\[ F^{(n)} \]
\[ \vdots \]
\[ F^{(3)} \]
\[ \vdots \]
\[ F^{(2)} \]
\[ \vdots \]
\[ F \]
Inflation maps along the Galois tower

Define Galois groups

$$G_F^{[n]} := \text{Gal}(F^{(n)}/F)$$
$$G_F^{(n)} := \text{Gal}(F^{(p)}/F^{(n)})$$

In fact

$$G_F^{(n)} = \left[ G_F, G_F^{(n-1)} \right] \left( G_F^{(n-1)} \right)^p.$$

These fit in a sequence

$$1 \to G_F^{(n)} \to G_F \to G_F^{[n]} \to 1.$$

Inflation maps: $$\text{inf} : H^*(G_F^{[n]}, \mathbb{F}_p) \longrightarrow H^*(G_F^{[n+1]}, \mathbb{F}_p)$$
Lemma

The inflation map \( \text{inf}: H^1(G_F^{[2]}, \mathbb{F}_p) \to H^1(G_F, \mathbb{F}_p) \) is a natural isomorphism.

Proof.
Since \( H^1(G, \mathbb{F}_p) \cong \text{Hom}(G, \mathbb{F}_p) \), the inflation map in question sends \( f: G_F^{[2]} \to \mathbb{F}_p \) to \( f\pi: G_F \to \mathbb{F}_p \), where \( \pi: G_F \twoheadrightarrow G_F^{[2]} \) is the quotient map. Injectivity is now clear.

Surjectivity follows from the fact that commutators and \( p \)-th powers belong to the kernel of every map \( G_F \to \mathbb{F}_p \).

Thus \( H^1(G_F, \mathbb{F}_p) \) can be captured at the level of \( F^{(2)} \).
Theorem (C.–Mináč, 2007)

The decomposable part of $H^*(G_F[3], \mathbb{F}_p)$ is naturally isomorphic to $H^*(G_F, \mathbb{F}_p)$.

Proof Summary: The isomorphism is given by the inflation map

$$\text{inf}: \text{Dec}(H^*(G_F[3], \mathbb{F}_p)) \longrightarrow H^*(G_F, \mathbb{F}_p).$$
The second cohomology refinement of Bloch–Kato

**Theorem (C.–Mináč, 2007)**

Let \( n \geq 3 \). Then we have the following:

1. **The inflation map** \( \text{inf} : H^2(G_F^{[n]}, \mathbb{F}_p) \longrightarrow H^2(G_F^{[n+1]}, \mathbb{F}_p) \) sends **indecomposable** classes to **decomposable** classes.

2. \( F(n^{+1}) \) is the smallest field extension which decomposes all classes in \( H^2(G_F^{[n]}, \mathbb{F}_p) \).

3. \( F(n^{+1}) = F(n)[J_n^{1/p}] \), where \( J_n \cong \left( F(n)^* / F(n)^*p \right)^{G_F^{[n]}} \).

Moreover,

\[
J_n \cong \ker[\text{inf} : H^2(G_F^{[n]}, \mathbb{F}_p) \longrightarrow H^2(G_F, \mathbb{F}_p)].
\]
We are investigating how the indecomposable classes in higher cohomology at various levels in the Galois Tower decompose under the inflation maps.

One know by the Bloch–Kato conjecture that they decompose completely into one–dimensional classes when we inflate them all the way up to $F(p)$.

WHAT HAPPENS IN BETWEEN?? This is the mystery, if revealed, which will give a refinement of the Bloch–Kato conjecture!

This is work in progress with Benson and Swallow.
New interpretation of maximal $p$–extensions

Theorem (C.–Mináč, 2007)

Let $H$ be a proper quotient of $G_F$ such that

$$G_F \twoheadrightarrow H \twoheadrightarrow G_F^{[3]}.$$ 

Then $H^2(H, \mathbb{F}_p)$ contains an non-zero indecomposable class.

**Moral:** The maximal pro-$p$-extension $F(p)$ is the smallest subfield of the separable closure (which contains $F^{(3)}$) and whose Galois cohomology is generated by one–dimensional classes!
Question: Given a pro-
$p$-
 group \( G \), when is it an absolute Galois group?

The previous theorem allows us to eliminate pro-
$p$-
 groups which can be Galois groups of maximal \( p \)-extensions.

Suppose if \( G \) is a pro-
$p$-
 group such that

\[
G \rightarrow H \rightarrow G^{[3]},
\]

and cohomology of \( H \) is decomposable (eg. free products of Demuškin groups), then \( G \) cannot be a Galois group of a maximal \( p \)-extension.

A pro–\( p \)–group \( D \) is a Demuškin group if it has cohomological dimension 2 and Poincaré duality:

\[
\langle \ , \rangle : \ H^1(D, \mathbb{F}_p) \times H^1(D, \mathbb{F}_p) \xrightarrow{\cup} H^2(D, \mathbb{F}_p) = \mathbb{F}_p
\]
Thank You