A new perspective on groups with periodic cohomology

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Outline

- Groups with periodic cohomology
- Stable module category and Tate cohomology
- The finite generation problem
- Some related results
Let $G$ be a finite group and let $p$ be a prime.

The group cohomology $H^*(G, \mathbb{F}_p)$ of $G$ is just the ordinary mod–$p$ cohomology of the classifying space $BG$ of $G$.

We say $G$ has periodic cohomology if for some integer $d > 0$

$$H^{*+d}(G, \mathbb{F}_p) \cong H^*(G, \mathbb{F}_p)$$

**Artin–Tate** (1952): $G$ has periodic cohomology if and only if the Sylow $p$–group of $G$ is either a cyclic group or a generalised Quaternion group.

**Swan** (1960): $G$ has periodic cohomology for all $p$ if and only if $G$ acts freely on a finite CW complex which has the homotopy type of a sphere.
The stable module category

$G$ is a finite group and $k$ is a field of characteristic $p > 0$ which divides the order of $G$.

$f, g: M \to N$ are homotopic if their difference $f - g$ factors through a projective.

The stable module category – stmod$(kG)$

- Objects: finitely generated $kG$–modules.
- Morphisms: homotopy classes of $kG$–linear maps.

stmod$(kG)$ is a triangulated category:

1. $\Omega(M) := \ker(P_M \to M)$
2. exact triangles are given by the short exact sequences of $kG$–modules.
Tate cohomology functor:

\[ \hat{H}^*(G, -) : \text{stmod}(kG) \rightarrow \text{graded vector spaces over } k. \]

\[ M \leftrightarrow \hat{H}^*(G, M) \cong \text{Hom}_{kG}(\Omega^* k, M). \]

Tate cohomology ring of \( G \) is \( \hat{H}^*(G, k) \). This is a graded commutative ring. Cup products are given by composition of maps!

\( \hat{H}^*(G, M) \) is naturally a module over \( \hat{H}^*(G, k) \).

Tate cohomology in positive degrees is just the ordinary cohomology.

Tate duality: Ties up positive and negative cohomologies

\[ \langle \ , \rangle : \hat{H}^i(G, k) \otimes \hat{H}^{-i-1}(G, k) \rightarrow \hat{H}^{-1}(G, k) \cong k \]
The finite generation problem

Let $M$ be a finitely–generated $kG$–module.


Question: Is it true that $\hat{H}^*(G, M)$ is finitely generated as a graded module over $\hat{H}^*(G, k)$?

Answer: resounding NO!

Weaker question: Which groups $G$ have the property that $\hat{H}^*(G, M)$ is finitely generated for all $M$?

Partial Answer: Groups with periodic cohomology have this property. (Proof is Clear.)
Theorem (Carlson–C–Minac, 2007)
\( \hat{\mathbb{H}}^*(G, M) \) is finitely generated for all finitely generated \( kG \)-modules \( M \) if and only if \( G \) has periodic cohomology.

Combining this with Swan’s result, we get:

Corollary (Carlson–C–Minac, 2007)
\( \hat{\mathbb{H}}^*(G, M) \) is finitely generated for all \( M \) and all \( p \) if and only if \( G \) acts freely on a finite CW complex which has the homotopy type of a sphere.
Proof Strategy

It suffices to show that if $G$ has non–periodic cohomology then there exists a $kG$–module whose Tate cohomology is not finitely generated.

We proved a much stronger statement:

**Theorem (Carlson–C–Minac, 2007)**

Suppose that $G$ has non–periodic cohomology. Let $M$ be any non–projective periodic $kG$–module. Then $\hat{H}^*(G, \text{End}_k M)$ is not finitely generated as a $\hat{H}^*(G, k)$-module.

$M$ is periodic if $\Omega^t M \cong M$ in $\text{stmod}(kG)$ for some $t$.

Proof uses Chouinard’s theorem, cyclic shifted subgroups, a result on negative cohomology by Benson and Carlson.
Chouinard (1976): There exists a maximal elementary abelian $p$–subgroup $E = \langle x_1, \ldots, x_n \rangle$ such that the restriction $M_E$ is not projective.

There exists an element $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$ and a corresponding cyclic shifted subgroup $\langle u_\alpha \rangle$,

$$u_\alpha := 1 + \sum_{i=1}^{n} \alpha_i (x_i - 1)$$

such that the restriction of $M$ to $\langle u_\alpha \rangle$ is not projective.

Cohomological interpretation: The identity map $\text{Id}_M : M \longrightarrow M$ does not factor through a projective $k \langle u_\alpha \rangle$–module. $\hat{H}^0 (\langle u_\alpha \rangle, \text{End}_k M) \cong \text{Hom}_{k \langle u_\alpha \rangle} (k, \text{End}_k M)$, so we have a non–zero class in $\hat{H}^0$ which is represented by $1 \in k \mapsto \text{Id}_M$. 

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### A new perspective on groups with periodic cohomology
Two crucial ingredients

Let $t$ be the period of $M$ so that $\Omega^{mt}(M) \cong M$ for all $m$.

I. By periodicity there exists elements

$$\zeta_m \in \hat{H}^{mt}(G, \text{End}_k M)$$

such that $\zeta_m$ is not zero on restriction to $\langle u_\alpha \rangle$.

II. Benson–Carlson (1992): Since $G$ has $p$–rank at least 2, the restriction map

$$\text{res}_{G,\langle u_\alpha \rangle} : \hat{H}^d(G, k) \longrightarrow \hat{H}^d(\langle u_\alpha \rangle, k)$$

is the zero map if $d < 0$. 
Suppose that $\hat{H}^\ast (G, \text{End}_k M)$ is finitely generated.

Then there exist generators $\mu_1, \ldots, \mu_r$ in degrees $d_1, \ldots, d_r$, respectively, in $\hat{H}^\ast (G, \text{End}_k M)$.

Now choose an integer $m$ such that $mt < \min\{d_i\}$.

Then we must have that

$$\zeta_m = \sum_{i=1}^{r} \gamma_i \mu_i$$

for some $\gamma_i \in \hat{H}^{mt - d_i} (G, k)$.

I. $\implies \text{res}_{G, \langle u_\alpha \rangle} (\zeta_m) \neq 0$.

II. $\implies \text{res}_{G, \langle u_\alpha \rangle} (\gamma_i) = 0 \ \forall \ i \ (\text{since } mt - d_i < 0)$.

QED.
Definition: The support variety $V_G(M)$ of $M$ is defined as follows.

$$V_G(M) := \text{Var} \left[ \ker : \text{Ext}^*_k(k, k) \otimes^M \text{Ext}^*_k(M, M) \right]$$

Support varieties are closed homogeneous subvarieties of the maximal ideal spectrum for the group cohomology ring.

Conjecture: For an indecomposable non-projective $kG$–module $M$, $\hat{H}^*(G, M)$ is finitely generated over $\hat{H}^*(G, k)$ if and only if

$$V_G(M) = V_G(k).$$
The following result supports our conjecture.

**Theorem (Carlson–C.–Mináč, 2007)**

Let $G$ be a group with the property that product of any two elements in negative cohomology is trivial. If $V_G(M) \subseteq V_G(\zeta)$ for some regular element $\zeta$ in the cohomology ring, then $\hat{H}^*(G, M)$ is not finitely generated over $\hat{H}^*(G, k)$.

**Example:** For $G$ as above, let $\zeta$ be a regular element in $H^*(G, k)$.

$$L_\zeta := \text{Ker}(\zeta : \Omega^{||\zeta||} \rightarrow k)$$

$V_G(L_\zeta) = V_G(\zeta)$ and consequently $\hat{H}^*(G, L_\zeta)$ is not finitely generated.
Theorem (Carlson–C.–Mináč, 2007)

Let $G$ be a group with non–periodic cohomology. Let $\zeta$ be an element in the Tate cohomology ring of $G$ such that

$$\dim \dim \text{Im}[\zeta : \hat{H}^\ast(G, k) \to \hat{H}^\ast(G, k)] < \infty$$

Then the Tate cohomology of the module $M$ represented by $\zeta$ is finitely generated over $H^\ast(G, k)$.

If $\zeta$ belongs to $\hat{H}^{n+1}(G, k)$, then there exists a module $M$ which sits in:

$$\zeta : 0 \to k \to M \to \Omega^n k \to 0$$

The middle terms of Auslander–Reiten sequences ending in $k$ are a good source of examples.
Thank You