The Unreasonably Beautiful World of Numbers

Sunil K. Chebolu
Illinois State University

Presentation for Math Club, March 3rd, 2010
Why are numbers beautiful? It's like asking why is Beethoven’s Ninth Symphony beautiful. If you don’t see why, someone can’t tell you. I know numbers are beautiful. If they aren’t beautiful, nothing is.

– Paul Erdös
Outline

- Hindu-Arabic numerals
- Tricks and puzzles with numbers
- Some important landmarks in number theory
- Open problems
The Hindu-Arabic numerals

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

The Hindu-Arabic numerals is a positional decimal number system.

These were introduced by the Indian mathematicians in the 8th century.

The Arabs, however, played an essential part in the dissemination of this numeral system.
The concept of *shunya* originated in the Vedas in ancient India has a long history and varied manifestations in different dimensions, in mathematics, in philosophy and in mysticism.

In mathematical literature it is used in the sense of ‘zero’ having no numerical value of its own but playing the key role in the system of decimal notation (*dasa*).

This discovery of shunya (a symbol for nothing) and the place value system were unique to the Indian civilization. *These ideas have escaped some of the greatest minds of antiquity, including Archimedes.*
<table>
<thead>
<tr>
<th>Brahmi</th>
<th>० १ २ ३ ४ ५ ६ ७ ८ ९</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hindu</td>
<td>० १ २ ३ ४ ५ ६ ७ ८ ९</td>
</tr>
<tr>
<td>Arabic</td>
<td>0 1 2 3 4 5 6 7 8 9</td>
</tr>
<tr>
<td>Medieval</td>
<td>0 1 2 3 8 9 6 7 8 9</td>
</tr>
<tr>
<td>Modern</td>
<td>0 1 2 3 4 5 6 7 8 9</td>
</tr>
</tbody>
</table>
Why the number “one” is “1” and the number “two” is “2” etc.? Where did these symbols come from?

Here is the answer ...
A mind reading trick

1. Choose a single digit number (not zero).
3. If the answer has two digits, add them.
4. Subtract 5 from what you have.
5. Turn this number into a letter by the rule $A = 1$, $B = 2$, $C = 3$, and so on.
6. Think of a country beginning with this letter.
7. Finally, take the last letter of this country and think of an animal beginning with that letter.
How does this work?

Look at the multiples of 9:

9, 18, 27, 36, 45, 54, 63, 72, 81

The digits of these numbers always add up to 9.

Therefore when you subtract a 5 in step 4, you will always get a 4.

There are not many countries that start with the letter D. The first country that comes to mind is Denmark.

There are not many animals with starting letter K. Kangaroo is probably the only one.

However, one other possibility is Dominican Republic, which gives Cat in the final step of the trick. But that is very rare!
A trick with match sticks

A magician claims that he can tell how many matches there are in a box just by listening to the rattle of the contents.

1. You (the magician) hand a match box to a member of the audience. The match box contains a known number of matches – 29 is a good number, as you will see.

2. The audience participant is asked to take the matches out and replace as many as they wish, counting as they go.

3. You ask them to add the digits of this number and remove those many matches from the box. Then they return the box to you.

4. You then shake the box and tell exactly how many matches remain.
How does this work?

One line answer: The test of divisibility for the number 9.

Every number is congruent modulo 9 to the sum of its digits. This means, a number $N$ minus the sum of the digits of $N$ is always divisible by 9.

For a match box with at most 29 matches in it, the number of matches left in the match box can only be 9 or 18.

With a little practice, it is not hard to tell whether the number is 9 or 18 by listening to the rattle carefully.

In fact, you can try a similar trick based on the test of divisibility for the number 11.
A matrix puzzle

Fill the matrix below with integers such that the sum of any five integers which do not belong to the same row or column is 57

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
How do we construct such a matrix?

Answer: Really simple.

Start with any 10 numbers which add up to 57. Say,

\[12, 1, 4, 18, 0, 7, 0, 4, 9, 2\]

\[
\begin{array}{cccccc}
12 & 1 & 4 & 18 & 0 \\
7 & 19 & 8 & 11 & 25 & 7 \\
0 & 12 & 1 & 4 & 18 & 0 \\
4 & 16 & 5 & 8 & 22 & 4 \\
9 & 21 & 10 & 13 & 27 & 9 \\
2 & 14 & 3 & 6 & 20 & 2 \\
\end{array}
\]

Addition Table
Circle any 5 numbers in the above addition matrix such that no two belong to the same row or column. What is the sum of these 5 numbers?

\[
\begin{array}{ccccc}
7 & 19 & 8 & 11 & 25 & 7 \\
0 & 12 & 1 & 4 & 18 & 0 \\
4 & 16 & 5 & 8 & 22 & 4 \\
9 & 21 & 10 & 13 & 27 & 9 \\
2 & 14 & 3 & 6 & 20 & 2 \\
\end{array}
\]

\[
7 + 1 + 22 + 13 + 14 = 57
\]
Circular sudoku

Fill in the grid so that every ring and every pair of neighboured circle segments contains the digits 1 through 8.

Invented by Professor Peter Higgins, at Essex University, UK.
The Taxicab number

Hardy: The number of my taxicab was 1729. It seemed to me rather a dull number.

Ramanujan: No, Hardy! No! It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways.

\[ 1729 = 1^3 + 12^3 = 9^3 + 10^3 \]
Since then, integer solutions to

\[ I^3 + J^3 = K^3 + L^3 \]

have been called Ramanujan Numbers.

The first five of these are:

<table>
<thead>
<tr>
<th>I</th>
<th>J</th>
<th>K</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
<td>9</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>18</td>
<td>20</td>
</tr>
<tr>
<td>10</td>
<td>27</td>
<td>19</td>
<td>24</td>
</tr>
<tr>
<td>4</td>
<td>32</td>
<td>18</td>
<td>30</td>
</tr>
</tbody>
</table>
1. Pick a positive integer $n$
2. If $n$ is even divide by 2, and if $n$ is odd replace it with $3n + 1$
3. repeat this process

Lothar Collatz in 1937 conjectured that this process will eventually reach the number 1.

For instance, starting with $n = 7$, we get

$$7 \rightarrow 22 \rightarrow 11 \rightarrow 34 \rightarrow 17 \rightarrow 52 \rightarrow 26 \rightarrow 13 \rightarrow 40 \rightarrow 20$$
$$\rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
This conjecture is wide open although a lot of work has been done on it. It has been verified by computers for $n$ up to a million million!

For instance, it is known that of the first 1000 integers more than 350 have a hailstone maximum height of 9232 before collapsing to 1.

Paul Erdős (1913 - 1996) regarded this as a problem for which ‘mathematics was not ready yet.’
The irrational numbers

The most unexpected theorem in mathematics:

\[ \sqrt{2} \text{ is irrational.} \]

The Greeks were quite disturbed to discover this fact. For, it has ruined the famous slogan of the Pythagorean school:

“All is number.”
Fermat’s Last Theorem states that the equation

\[ x^n + y^n = z^n \]

has no positive integer solutions for \( x, y \) and \( z \) when \( n \geq 3 \).

Fermat, in his copy of Diophantus’s Arithmetica, wrote in 1637 what became the most enigmatic note in the history of mathematics: “I have discovered a truly remarkable proof which this margin is too small to contain.”
FLT was proved in 1995 by Sir Andrew Wiles from Princeton University. Professor Wiles proved the Taniyama-Shimura conjecture which implies FLT.
Theorem: $2^{1/n}$ is irrational if $n \geq 3$

Proof: Suppose to the contrary $2^{1/n}$ is rational. Let $2^{1/n} = \frac{a}{b}$.

Then we have

$$2 = \frac{a^n}{b^n}.$$

This gives

$$2b^n = a^n.$$

From there we get

$$b^n + b^n = a^n.$$

A contradiction to Fermat’s last theorem. QED.

Mathematics is consistent!
Why is the standard $A_4$ size paper 210 mm x 297 mm?

$$\frac{297}{210} - \sqrt{2} \approx 0.000072152$$

The dimensions of the $A_4$ paper are chosen from a good rational approximation of $\sqrt{2}$.

But why chose $\sqrt{2}$?
A rectangular paper when folded down the middle on its longer side should give two smaller sheets which are similar to the original sheet.

Suppose the sides of a rectangular sheet have lengths $a$ and $b$ ($b > a$).

$$a : b = \frac{b}{2} : a$$

This means

$$\frac{a}{b} = \frac{b}{2a} \implies 2a^2 = b^2 \implies \sqrt{2} = \frac{b}{a}.$$
The Goldbach conjecture: Every even integer greater than 2 is a sum of two primes.

The Twin prime conjecture: There exist infinitely many twin primes?

The Catalan conjecture: The only non-trivial solution in positive integers for the equation $x^y - y^x = 1$ is $x = 3$ and $y = 2$.

Primes in arithmetic progression: Are there arbitrarily long arithmetic progressions consisting of prime numbers?
Wir müssen wissen
Wir werden wissen