1. For a special whole life insurance on \((x)\), payable at the moment of death:

(i) \(\mu_x(t) = 0.05\), \(t > 0\)

(ii) \(\delta = 0.08\)

(iii) The death benefit at time \(t\) is \(b_t = e^{0.06t}\), \(t > 0\).

(iv) \(Z\) is the present value random variable for this insurance at issue.

Calculate \(\text{Var}(Z)\).

(A) 0.038
(B) 0.041
(C) 0.043
(D) 0.045
(E) 0.048
2. For a group of individuals all age $x$, you are given:

(i) 25% are smokers (s); 75% are nonsmokers (ns).

(ii)

<table>
<thead>
<tr>
<th>$k$</th>
<th>$q_{x+k}^s$</th>
<th>$q_{x+k}^{ns}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.10</td>
<td>0.05</td>
</tr>
<tr>
<td>1</td>
<td>0.20</td>
<td>0.10</td>
</tr>
<tr>
<td>2</td>
<td>0.30</td>
<td>0.15</td>
</tr>
</tbody>
</table>

(iii) $i = 0.02$

Calculate $10,000 A_x^{12}$ for an individual chosen at random from this group.

(A) 1690
(B) 1710
(C) 1730
(D) 1750
(E) 1770
3. For a fully continuous whole life insurance of 1 on \((x)\), you are given:

(i) The forces of mortality and interest are constant.

(ii) \(\frac{2}{\overline{A}_x} = 0.20\)

(iii) \(\overline{P}(\overline{A}_x) = 0.03\)

(iv) \(L\) is the loss-at-issue random variable based on the benefit premium.

Calculate \(\text{Var}(L)\).

(A) 0.20

(B) 0.21

(C) 0.22

(D) 0.23

(E) 0.24
4. For a population which contains equal numbers of males and females at birth:

(i) For males, $\mu_m(x) = 0.10, \quad x \geq 0$

(ii) For females, $\mu_f(x) = 0.08, \quad x \geq 0$

Calculate $q_{60}$ for this population.

(A) 0.076
(B) 0.081
(C) 0.086
(D) 0.091
(E) 0.096
5. You are simulating the future lifetimes of newborns in a population.

(i) For any given newborn, mortality follows De Moivre’s law with maximum lifetime $\Omega$.

(ii) $\Omega$ has distribution function $F(\omega) = (\omega/80)^2$, $0 \leq \omega \leq 80$.

(iii) You are using the inverse transform method, with small random numbers corresponding to small values of $\Omega$ or short future lifetimes.

(iv) Your first random numbers from $[0,1]$ for simulating $\Omega$ and the future lifetime are 0.4 and 0.7 respectively.

Calculate your first simulated value of the future lifetime.

(A) 22
(B) 35
(C) 46
(D) 52
(E) 56
6. You are simulating a compound claims distribution:

(i) The number of claims, \( N \), is binomial with \( m = 3 \) and mean 1.8.

(ii) Claim amounts are uniformly distributed on \( \{1, 2, 3, 4, 5\} \).

(iii) Claim amounts are independent, and are independent of the number of claims.

(iv) You simulate the number of claims, \( N \), then the amounts of each of those claims, \( X_1, X_2, \ldots, X_N \). Then you repeat another \( N \), its claim amounts, and so on until you have performed the desired number of simulations.

(v) When the simulated number of claims is 0, you do not simulate any claim amounts.

(vi) All simulations use the inverse transform method, with low random numbers corresponding to few claims or small claim amounts.

(vii) Your random numbers from (0, 1) are 0.7, 0.1, 0.3, 0.1, 0.9, 0.5, 0.5, 0.7, 0.3, and 0.1.

Calculate the aggregate claim amount associated with your third simulated value of \( N \).

(A) 3

(B) 5

(C) 7

(D) 9

(E) 11
7. Annual prescription drug costs are modeled by a two-parameter Pareto distribution with \( \theta = 2000 \) and \( \alpha = 2 \).

A prescription drug plan pays annual drug costs for an insured member subject to the following provisions:

(i) The insured pays 100% of costs up to the ordinary annual deductible of 250.
(ii) The insured then pays 25% of the costs between 250 and 2250.
(iii) The insured pays 100% of the costs above 2250 until the insured has paid 3600 in total.
(iv) The insured then pays 5% of the remaining costs.

Determine the expected annual plan payment.

(A) 1120
(B) 1140
(C) 1160
(D) 1180
(E) 1200
8. For a tyrannosaur with a taste for scientists:

(i) The number of scientists eaten has a binomial distribution with \( q = 0.6 \) and \( m = 8 \).

(ii) The number of calories of a scientist is uniformly distributed on \((7000, 9000)\).

(iii) The numbers of calories of scientists eaten are independent, and are independent of the number of scientists eaten.

Calculate the probability that two or more scientists are eaten and exactly two of those eaten have at least 8000 calories each.

(A) 0.23

(B) 0.25

(C) 0.27

(D) 0.30

(E) 0.35
9. For a special fully continuous last survivor insurance of 1 on $x$ and $y$, you are given:

(i) $T(x)$ and $T(y)$ are independent.

(ii) $\mu_x(t) = 0.08$, $t > 0$

(iii) $\mu_y(t) = 0.04$, $t > 0$

(iv) $\delta = 0.06$

(v) $\pi$ is the annual benefit premium payable until the first of $x$ and $y$ dies.

Calculate $\pi$.

(A) 0.055

(B) 0.080

(C) 0.105

(D) 0.120

(E) 0.150
10. For a special fully discrete whole life insurance of 1000 on (42):

(i) The contract premium for the first 4 years is equal to the level benefit premium for a fully discrete whole life insurance of 1000 on (40).

(ii) The contract premium after the fourth year is equal to the level benefit premium for a fully discrete whole life insurance of 1000 on (42).

(iii) Mortality follows the Illustrative Life Table.

(iv) \( i = 0.06 \)

(v) \( 3L \) is the prospective loss random variable at time 3, based on the contract premium.

(vi) \( K(42) \) is the curtate future lifetime of (42).

Calculate \( E[3L|K(42) \geq 3] \).

(A) 27
(B) 31
(C) 44
(D) 48
(E) 52
11. Your company is competing to sell a life annuity-due with an actuarial present value of 500,000 to a 50-year old individual.

Based on your company’s experience, typical 50-year old annuitants have a complete life expectancy of 25 years. However, this individual is not as healthy as your company’s typical annuitant, and your medical experts estimate that his complete life expectancy is only 15 years.

You decide to price the benefit using the issue age that produces a complete life expectancy of 15 years. You also assume:

(i) For typical annuitants of all ages, mortality follows De Moivre’s Law with the same limiting age, $\omega$.

(ii) $i = 0.06$

Calculate the annual benefit that your company can offer to this individual.

(A) 38,000

(B) 41,000

(C) 46,000

(D) 49,000

(E) 52,000
12. For a double decrement table, you are given:

(i) \( q_x^{(1)} = 0.2 \)

(ii) \( q_x^{(2)} = 0.3 \)

(iii) Each decrement is uniformly distributed over each year of age in the double decrement table.

Calculate \( 0.3q_x^{(1)} \).

(A) 0.020

(B) 0.031

(C) 0.042

(D) 0.053

(E) 0.064
13-14. Use the following information for questions 13 and 14.

For a Markov model for an insured population:

(i) Annual transition probabilities between health states of individuals are as follows:

<table>
<thead>
<tr>
<th></th>
<th>Healthy</th>
<th>Sick</th>
<th>Terminated</th>
</tr>
</thead>
<tbody>
<tr>
<td>Healthy</td>
<td>0.7</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>Sick</td>
<td>0.3</td>
<td>0.6</td>
<td>0.1</td>
</tr>
<tr>
<td>Terminated</td>
<td>0.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

(ii) The mean annual healthcare cost each year for each health state is:

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Healthy</td>
<td>500</td>
</tr>
<tr>
<td>Sick</td>
<td>3000</td>
</tr>
<tr>
<td>Terminated</td>
<td>0</td>
</tr>
</tbody>
</table>

(iii) Transitions occur at the end of the year.

(iv) $i = 0$

13. Calculate the expected future healthcare costs (including the current year) for an insured individual whose current state is healthy.

Recall:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \begin{pmatrix} a_{22} / d & -a_{12} / d \\ -a_{21} / d & a_{11} / d \end{pmatrix}$$

where $d = a_{11}a_{22} - a_{12}a_{21}$

(A) 5100
(B) 5600
(C) 6100
(D) 6600
(E) 7100
13-14. (Repeated for convenience) Use the following information for questions 13 and 14.

For a Markov model for an insured population:

(i) Annual transition probabilities between health states of individuals are as follows:

<table>
<thead>
<tr>
<th></th>
<th>Healthy</th>
<th>Sick</th>
<th>Terminated</th>
</tr>
</thead>
<tbody>
<tr>
<td>Healthy</td>
<td>0.7</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>Sick</td>
<td>0.3</td>
<td>0.6</td>
<td>0.1</td>
</tr>
<tr>
<td>Terminated</td>
<td>0.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

(ii) The mean annual healthcare cost each year for each health state is:

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Healthy</td>
<td>500</td>
</tr>
<tr>
<td>Sick</td>
<td>3000</td>
</tr>
<tr>
<td>Terminated</td>
<td>0</td>
</tr>
</tbody>
</table>

(iii) Transitions occur at the end of the year.

(iv) \( i = 0 \)

14. A contract premium of 800 is paid each year by an insured not in the terminated state.

Calculate the expected value of contract premiums less healthcare costs over the first 3 years for a new healthy insured.

(A) –390
(B) –200
(C) –20
(D) 160
(E) 340
15. Two types of insurance claims are made to an insurance company. For each type, the number of claims follows a Poisson distribution and the amount of each claim is uniformly distributed as follows:

<table>
<thead>
<tr>
<th>Type of Claim</th>
<th>Poisson Parameter $\lambda$ for Number of Claims</th>
<th>Range of Each Claim Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>12</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>II</td>
<td>4</td>
<td>(0, 5)</td>
</tr>
</tbody>
</table>

The numbers of claims of the two types are independent and the claim amounts and claim numbers are independent.

Calculate the normal approximation to the probability that the total of claim amounts exceeds 18.

(A) 0.37
(B) 0.39
(C) 0.41
(D) 0.43
(E) 0.45
16. For a water reservoir:

(i) The present level is 4999 units.

(ii) 1000 units are used uniformly daily.

(iii) The only source of replenishment is rainfall.

(iv) The number of rainfalls follows a Poisson process with \( \lambda = 0.2 \) per day.

(v) The distribution of the amount of a rainfall is as follows:

<table>
<thead>
<tr>
<th>Amount</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>8000</td>
<td>0.2</td>
</tr>
<tr>
<td>5000</td>
<td>0.8</td>
</tr>
</tbody>
</table>

(vi) The numbers and amounts of rainfalls are independent.

Calculate the probability that the reservoir will be empty sometime within the next 10 days.

(A) 0.27

(B) 0.37

(C) 0.39

(D) 0.48

(E) 0.50
17. The number of annual losses has a Poisson distribution with a mean of 5. The size of each loss has a two-parameter Pareto distribution with $\theta = 10$ and $\alpha = 2.5$. An insurance for the losses has an ordinary deductible of 5 per loss.

Calculate the expected value of the aggregate annual payments for this insurance.

(A) 8  
(B) 13  
(C) 18  
(D) 23  
(E) 28
18. Losses in 2003 follow a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 5$. Losses in 2004 are uniformly 20% higher than in 2003. An insurance covers each loss subject to an ordinary deductible of 10.

Calculate the Loss Elimination Ratio in 2004.

(A) $\frac{5}{9}$

(B) $\frac{5}{8}$

(C) $\frac{2}{3}$

(D) $\frac{3}{4}$

(E) $\frac{4}{5}$
19. A company insures a town for liability during its annual parade. The company will charge an annual premium equal to expected claims plus a relative security load of 1/3.

You are given:

(i) Annual claims are 3, 5, or 7 and are independent.
(ii) \( p(3) = 0.75 \)
(iii) \( p(5) = 0.15 \)
(iv) Premiums are payable at the beginning of the year.
(v) \( i = 0 \)
(vi) Initial surplus = 3

Calculate the probability of ruin within the first two years.

(A) 0.01
(B) 0.02
(C) 0.05
(D) 0.08
(E) 0.10
20. The mortality of (x) and (y) follows a common shock model with components $T^*(x)$, $T^*(y)$ and $Z$.

(i) $T^*(x)$, $T^*(y)$ and $Z$ are independent and have exponential distributions with respective forces $\mu_1$, $\mu_2$ and $\lambda$.

(ii) The probability that (x) survives 1 year is 0.96.

(iii) The probability that (y) survives 1 year is 0.97.

(iv) $\lambda = 0.01$

Calculate the probability that both (x) and (y) survive 5 years.

(A) 0.65

(B) 0.67

(C) 0.70

(D) 0.72

(E) 0.74
21. For a fully discrete whole life insurance of 100,000 on each of 10,000 lives age 60, you are given:

(i) The future lifetimes are independent.

(ii) Mortality follows the Illustrative Life Table.

(iii) \( i = 0.06. \)

(iv) \( \pi \) is the premium for each insurance of 100,000.

Using the normal approximation, calculate \( \pi \), such that the probability of a positive total loss is 1%.

(A) 3340
(B) 3360
(C) 3380
(D) 3390
(E) 3400
22. For a special fully discrete 3-year endowment insurance on (75), you are given:

(i) The maturity value is 1000.
(ii) The death benefit is 1000 plus the benefit reserve at the end of the year of death.
(iii) Mortality follows the Illustrative Life Table.
(iv) $i = 0.05$

Calculate the level benefit premium for this insurance.

(A) 321
(B) 339
(C) 356
(D) 364
(E) 373
23. For a special fully discrete 3-year term insurance on (55), whose mortality follows a double decrement model:

(i) Decrement 1 is accidental death; decrement 2 is all other causes of death.

(ii) 

<table>
<thead>
<tr>
<th>$x$</th>
<th>$q_x^{(1)}$</th>
<th>$q_x^{(2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>0.002</td>
<td>0.020</td>
</tr>
<tr>
<td>56</td>
<td>0.005</td>
<td>0.040</td>
</tr>
<tr>
<td>57</td>
<td>0.008</td>
<td>0.060</td>
</tr>
</tbody>
</table>

(iii) $i = 0.06$

(iv) The death benefit is 2000 for accidental deaths and 1000 for deaths from all other causes.

(v) The level annual contract premium is 50.

(vi) $\underline{1}L$ is the prospective loss random variable at time 1, based on the contract premium.

(vii) $K(55)$ is the curtate future lifetime of (55).

Calculate $\mathbb{E}[\underline{1}L|K(55) \geq 1]$.

(A) 5
(B) 9
(C) 13
(D) 17
(E) 20
24. The future lifetime of (0) follows a two-parameter Pareto distribution with \( \theta = 50 \) and \( \alpha = 3 \).

Calculate \( \hat{e}_{20} \).

(A) 5
(B) 15
(C) 25
(D) 35
(E) 45
25. For a modeled insurance company:

(i) \( P \) is the continuous time, infinite horizon survival probability.

(ii) \( Q \) is the discrete time, infinite horizon survival probability.

(iii) \( R \) is the continuous time, finite horizon survival probability.

(iv) \( S \) is the discrete time, finite horizon survival probability.

Which of the following is always true?

(A) \( S \leq Q \)

(B) \( S \leq P \)

(C) \( Q \leq P \)

(D) \( R \leq P \)

(E) \( 2P \leq Q + R \)
26. Customers arrive at a store at a Poisson rate that increases linearly from 6 per hour at 1:00 p.m. to 9 per hour at 2:00 p.m.

Calculate the probability that exactly 2 customers arrive between 1:00 p.m. and 2:00 p.m.

(A) 0.016
(B) 0.018
(C) 0.020
(D) 0.022
(E) 0.024
27. For independent stochastic processes $X(t)$ and $Y(t)$:

(i) $X(t)$ is a Brownian motion process with $X(0) = 0$, drift coefficient $\mu = 0$, and variance parameter $\sigma^2 = 0.5$.

(ii) $Y(t)$ is a Brownian motion process with $Y(0) = 2$, drift coefficient $\mu = 0$, and variance parameter $\sigma^2 = 1$.

Calculate the probability that $X(t) \geq Y(t)$ for some $t$ in $[0, 5]$.

(A) 0.23
(B) 0.29
(C) 0.35
(D) 0.41
(E) 0.47
28. For (80) and (84), whose future lifetimes are independent:

<table>
<thead>
<tr>
<th>x</th>
<th>p_x</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>0.50</td>
</tr>
<tr>
<td>81</td>
<td>0.40</td>
</tr>
<tr>
<td>82</td>
<td>0.60</td>
</tr>
<tr>
<td>83</td>
<td>0.25</td>
</tr>
<tr>
<td>84</td>
<td>0.20</td>
</tr>
<tr>
<td>85</td>
<td>0.15</td>
</tr>
<tr>
<td>86</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Calculate the change in the value \(2 \mid q_{80:84} \) if \( p_{82} \) is decreased from 0.60 to 0.30.

(A) 0.03
(B) 0.06
(C) 0.10
(D) 0.16
(E) 0.19
29. At interest rate $i$:

(i) $\ddot{a}_x = 5.6$

(ii) The actuarial present value of a 2-year certain and life annuity-due of 1 on $(x)$ is $\ddot{a}_{x:2} = 5.6459$.

(iii) $e_x = 8.83$

(iv) $e_{x+1} = 8.29$

Calculate $i$.

(A) 0.077

(B) 0.079

(C) 0.081

(D) 0.083

(E) 0.084
30. For a deferred whole life annuity-due on (25) with annual payment of 1 commencing at age 60, you are given:

(i) Level benefit premiums are payable at the beginning of each year during the deferral period.

(ii) During the deferral period, a death benefit equal to the benefit reserve is payable at the end of the year of death.

Which of the following is a correct expression for the benefit reserve at the end of the 20th year?

(A) \left( \frac{\dd}{s_{35}} \right) s_{20}

(B) \left( s_{20} / \dd_{35} \right) s_{35}

(C) \left( s_{20} / \dd_{35} \right) s_{35}

(D) \left( s_{35} / \dd_{20} \right) s_{20}

(E) \left( \dd_{35} / s_{35} \right)
31. You are given:

(i) The future lifetimes of (50) and (50) are independent.

(ii) Mortality follows the Illustrative Life Table.

(iii) Deaths are uniformly distributed over each year of age.

Calculate the force of failure at duration 10.5 for the last survivor status of (50) and (50).

(A) 0.001
(B) 0.002
(C) 0.003
(D) 0.004
(E) 0.005
32. Bob is a carnival operator of a game in which a player receives a prize worth $W = 2^N$ if the player has $N$ successes, $N = 0, 1, 2, 3,…$. Bob models the probability of success for a player as follows:

(i) $N$ has a Poisson distribution with mean $\Lambda$.

(ii) $\Lambda$ has a uniform distribution on the interval $(0, 4)$.

Calculate $E[W]$.

(A) 5
(B) 7
(C) 9
(D) 11
(E) 13
33. You are simulating the gain/loss from insurance where:

(i) Claim occurrences follow a Poisson process with \( \lambda = \frac{2}{3} \) per year.

(ii) Each claim amount is 1, 2 or 3 with \( p(1) = 0.25, \quad p(2) = 0.25, \quad \text{and} \quad p(3) = 0.50 \).

(iii) Claim occurrences and amounts are independent.

(iv) The annual premium equals expected annual claims plus 1.8 times the standard deviation of annual claims.

(v) \( i = 0 \)

You use 0.75, 0.60, 0.40, and 0.20 from the unit interval to simulate time between claims, where small numbers correspond to longer times.

You use 0.30, 0.60, 0.20, and 0.70 from the unit interval to simulate claim size, where small numbers correspond to smaller claims.

Calculate the gain or loss during the first 2 years from this simulation.

(A) loss of 5

(B) loss of 4

(C) 0

(D) gain of 4

(E) gain of 5
34. Annual dental claims are modeled as a compound Poisson process where the number of claims has mean 2 and the loss amounts have a two-parameter Pareto distribution with $\theta = 500$ and $\alpha = 2$.

An insurance pays 80% of the first 750 of annual losses and 100% of annual losses in excess of 750.

You simulate the number of claims and loss amounts using the inverse transform method with small random numbers corresponding to small numbers of claims or small loss amounts.

The random number to simulate the number of claims is 0.8. The random numbers to simulate loss amounts are 0.60, 0.25, 0.70, 0.10 and 0.80.

Calculate the total simulated insurance claims for one year.

(A) 294
(B) 625
(C) 631
(D) 646
(E) 658
35. For a special whole life insurance:

(i) The benefit for accidental death is 50,000 in all years.

(ii) The benefit for non-accidental death during the first 2 years is return of the single benefit premium without interest.

(iii) The benefit for non-accidental death after the first 2 years is 50,000.

(iv) Benefits are payable at the moment of death.

(v) Force of mortality for accidental death: \( \mu_x^{(1)}(t) = 0.01, \quad t \geq 0 \)

(vi) Force of mortality for non-accidental death: \( \mu_x^{(2)}(t) = 2.29, \quad t \geq 0 \)

(vii) \( \delta = 0.10 \)

Calculate the single benefit premium for this insurance.

(A) 1,000

(B) 4,000

(C) 7,000

(D) 11,000

(E) 15,000
36. A special whole life insurance on $(x)$ pays 10 times salary if the cause of death is an accident and 500,000 for all other causes of death.

You are given:

(i) $\mu_x(r) (t) = 0.01, \ t \geq 0$

(ii) $\mu_x(\text{accident}) (t) = 0.001, \ t \geq 0$

(iii) Benefits are payable at the moment of death.

(iv) $\delta = 0.05$

(v) Salary of $(x)$ at time $t$ is $50,000 e^{0.04t}, \ t \geq 0$.

Calculate the actuarial present value of the benefits at issue.

(A) 78,000

(B) 83,000

(C) 92,000

(D) 100,000

(E) 108,000
37. \( Z \) is the present value random variable for a 15-year pure endowment of 1 on \((x)\):

(i) The force of mortality is constant over the 15-year period.

(ii) \( \nu = 0.9 \)

(iii) \( \text{Var}(Z) = 0.065 \text{E}[Z] \)

Calculate \( q_x \).

(A) 0.020

(B) 0.025

(C) 0.030

(D) 0.035

(E) 0.040
38. You are given:

(i) \( kV^A \) is the benefit reserve at the end of year \( k \) for type A insurance, which is a fully discrete 10-payment whole life insurance of 1000 on \((x)\).

(ii) \( kV^B \) is the benefit reserve at the end of year \( k \) for type B insurance, which is a fully discrete whole life insurance of 1000 on \((x)\).

(iii) \( q_{x+10} = 0.004 \)

(iv) The annual benefit premium for type B is 8.36.

(v) \( 10 V^A - 10 V^B = 101.35 \)

(vi) \( i = 0.06 \)

Calculate \( 11 V^A - 11 V^B \).

(A) 91
(B) 93
(C) 95
(D) 97
(E) 99
39. For a special fully discrete 3-year term insurance on \((x)\):

(i) \(b_{k+1} = \begin{cases} 
0 & \text{for } k = 0 \\
1,000(11-k) & \text{for } k = 1, 2
\end{cases}\)

(ii) \[
\begin{array}{c|c}
  k & q_{x+k} \\
  \hline
  0 & 0.200 \\
  1 & 0.100 \\
  2 & 0.097 \\
\end{array}
\]

(iii) \(i = 0.06\)

Calculate the level annual benefit premium for this insurance.

(A) 518  
(B) 549  
(C) 638  
(D) 732  
(E) 799
40. For a special 3-year temporary life annuity-due on \((x)\), you are given:

(i) 

<table>
<thead>
<tr>
<th>(t)</th>
<th>Annuity Payment</th>
<th>(P_{x+t})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>15</td>
<td>0.95</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>0.90</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>0.85</td>
</tr>
</tbody>
</table>

(ii) \(i = 0.06\)

Calculate the variance of the present value random variable for this annuity.

(A) 91  
(B) 102  
(C) 114  
(D) 127  
(E) 139

**END OF EXAMINATION**
# COURSE 3, Fall 2004
## Preliminary Answer Key

<table>
<thead>
<tr>
<th>Question #</th>
<th>Answer</th>
<th>Question #</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>D</td>
<td>21</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td>22</td>
<td>C</td>
</tr>
<tr>
<td>3</td>
<td>A</td>
<td>23</td>
<td>D</td>
</tr>
<tr>
<td>4</td>
<td>B</td>
<td>24</td>
<td>D</td>
</tr>
<tr>
<td>5</td>
<td>B</td>
<td>25</td>
<td>E</td>
</tr>
<tr>
<td>6</td>
<td>C</td>
<td>26</td>
<td>A</td>
</tr>
<tr>
<td>7</td>
<td>C</td>
<td>27</td>
<td>E</td>
</tr>
<tr>
<td>8</td>
<td>D</td>
<td>28</td>
<td>B</td>
</tr>
<tr>
<td>9</td>
<td>A</td>
<td>29</td>
<td>B</td>
</tr>
<tr>
<td>10</td>
<td>B</td>
<td>30</td>
<td>A</td>
</tr>
<tr>
<td>11</td>
<td>E</td>
<td>31</td>
<td>B</td>
</tr>
<tr>
<td>12</td>
<td>D</td>
<td>32</td>
<td>E</td>
</tr>
<tr>
<td>13</td>
<td>B</td>
<td>33</td>
<td>E</td>
</tr>
<tr>
<td>14</td>
<td>D</td>
<td>34</td>
<td>C</td>
</tr>
<tr>
<td>15</td>
<td>A</td>
<td>35</td>
<td>D</td>
</tr>
<tr>
<td>16</td>
<td>D</td>
<td>36</td>
<td>D</td>
</tr>
<tr>
<td>17</td>
<td>C</td>
<td>37</td>
<td>B</td>
</tr>
<tr>
<td>18</td>
<td>B</td>
<td>38</td>
<td>E</td>
</tr>
<tr>
<td>19</td>
<td>A</td>
<td>39</td>
<td>A</td>
</tr>
<tr>
<td>20</td>
<td>E</td>
<td>40</td>
<td>C</td>
</tr>
</tbody>
</table>
NOVEMBER 2004
COURSE 3 SOLUTIONS

Question #1
Key: D

\[ E[Z] = \int_0^\infty b_i v^i \mu(x + t) dt = \int_0^\infty e^{0.06t} e^{-0.08t} e^{-0.05t} \frac{1}{20} dt \]
\[ = \frac{1}{20} \left( \frac{100}{7} \right) \left[ -e^{-0.07t} \right]_0^\infty = \frac{5}{7} \]

\[ E[Z^2] = \int_0^\infty (b_i v^i)^2 \mu(x + t) dt = \int_0^\infty e^{0.12t} e^{-0.16t} e^{-0.05t} \frac{1}{20} dt = \frac{1}{20} \left[ e^{-0.09t} \right]_0^\infty = \frac{5}{9} \]

\[ Var[Z] = \frac{5}{9} - \left( \frac{5}{7} \right)^2 = 0.04535 \]

Question #2
Key: C

Let \( ns = \) nonsmoker and \( s = \) smoker

<table>
<thead>
<tr>
<th>( k )</th>
<th>( q^{(ns)}_{x+k} )</th>
<th>( p^{(ns)}_{x+k} )</th>
<th>( q^{(s)}_{x+k} )</th>
<th>( p^{(s)}_{x+k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.05</td>
<td>0.95</td>
<td>0.10</td>
<td>0.90</td>
</tr>
<tr>
<td>1</td>
<td>0.10</td>
<td>0.90</td>
<td>0.20</td>
<td>0.80</td>
</tr>
<tr>
<td>2</td>
<td>0.15</td>
<td>0.85</td>
<td>0.30</td>
<td>0.70</td>
</tr>
</tbody>
</table>

\[ A^{(ns)}_{x\geq2} = \nu q^{(ns)}_{x} + \nu^2 p^{(ns)}_{x} q^{(ns)}_{x+1} \]
\[ = \frac{1}{1.02} (0.05) + \frac{1}{1.02^2} (0.95 \times 0.10) = 0.1403 \]

\[ A^{(s)}_{x\geq2} = \nu q^{(s)}_{x} + \nu^2 p^{(s)}_{x} q^{(s)}_{x+1} \]
\[ = \frac{1}{1.02} (0.10) + \frac{1}{1.02^2} (0.90 \times 0.20) = 0.2710 \]

\[ A_{x\geq2} = \text{weighted average} = (0.75)(0.1403) + (0.25)(0.2710) = 0.1730 \]
Question #3  
**Key: A**

\[
\bar{P}(A_x) = \mu = 0.03
\]

\[2\bar{A}_x = 0.20 = \frac{\mu}{2\delta + \mu} = \frac{0.03}{2\delta + 0.03} \]

\[\Rightarrow \delta = 0.06 \]

\[\text{Var}(0_{\delta}L) = \frac{2\bar{A}_x - (\bar{A}_x)^2}{(\delta \bar{a})^2} = \frac{0.20 - \left(\frac{1}{3}\right)^2}{\left(\frac{0.06}{0.09}\right)^2} = 0.20 \]

where \[A = \frac{\mu}{\mu + \delta} = \frac{0.03}{0.09} = \frac{1}{3} \quad \bar{a} = \frac{1}{\mu + \delta} = \frac{1}{0.09} \]

---

**Question #4**  
**Key: B**

\[s(60) = \frac{e^{-0.1(60)} + e^{-0.08(60)}}{2} = 0.005354 \]

\[s(61) = \frac{e^{-0.1(61)} + e^{-0.08(61)}}{2} = 0.00492 \]

\[q_{60} = 1 - \frac{0.00492}{0.005354} = 0.081 \]
Question #5
Key: B

For $\Omega$, $0.4 = F(\omega) = \left(\frac{\omega}{80}\right)^2$

$0.6325 = \frac{\omega}{80}$

$\omega = 50.6$

For $T(0)$ using De Moivre, $0.7 = F(t) = \frac{t}{\omega} = \frac{t}{50.6}$

$t = (0.7)(50.6) = 35.42$

Question #6
Key: C

$E[N] = mq = 1.8 \Rightarrow q = \frac{1.8}{3} = 0.6$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f_N(x)$</th>
<th>$F_N(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.064</td>
<td>0.064</td>
</tr>
<tr>
<td>1</td>
<td>0.288</td>
<td>0.352</td>
</tr>
<tr>
<td>2</td>
<td>0.432</td>
<td>0.784</td>
</tr>
<tr>
<td>3</td>
<td>0.216</td>
<td>1.000</td>
</tr>
</tbody>
</table>

First: $0.432 < 0.7 < 0.784$ so $N = 2$. Use 0.1 and 0.3 for amounts
Second: $0.064 < 0.1 < 0.352$ so $N = 1$ Use 0.9 for amount
Third: $0.432 < 0.5 < 0.784$ so $N = 2$ Use 0.5 and 0.7 for amounts

Discrete uniform $\Rightarrow F_X(x) = 0.2x$, $x = 1, 2, 3, 4, 5$

$0.4 < 0.5 < 0.6 \Rightarrow x_1 = 3$

$0.6 < 0.7 < 0.8 \Rightarrow x_2 = 4$

Aggregate claims = $3 + 4 = 7$
Question #7
Key: C

\[ E(X \land x) = \frac{\theta}{\alpha - 1} \left[ 1 - \left( \frac{\theta}{x + \theta} \right)^{\alpha - 1} \right] = \frac{2000x}{x + 2000} \]

<table>
<thead>
<tr>
<th>x</th>
<th>( E(X \land x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>2000</td>
</tr>
<tr>
<td>250</td>
<td>222</td>
</tr>
<tr>
<td>2250</td>
<td>1059</td>
</tr>
<tr>
<td>5100</td>
<td>1437</td>
</tr>
</tbody>
</table>

\[ 0.75(\overline{E}(X \land 2250) - \overline{E}(X \land 250)) + 0.95(\overline{E}(X) - \overline{E}(X \land 5100)) \]

\[ 0.75(1059 - 222) + 0.95(2000 - 1437) = 1162.6 \]

The 5100 breakpoint was determined by when the insured’s share reaches 3600:

\[ 3600 = 250 + 0.75(2250 - 250) + (5100 - 2250) \]

Question #8
Key: D

Since each time the probability of a heavy scientist is just half the probability of a success, the distribution is binomial with \( q = 0.6 \times 0.5 = 0.3 \) and \( m = 8 \).

\[ f(2) = (8 \times \frac{7}{2}) \times (0.3^2) \times (0.7^6) = 0.30 \]
Question #9
Key: A

\[ \mu_{xy}(t) = \mu_x(t) + \mu_y(t) = 0.08 + 0.04 = 0.12 \]
\[ \overline{A}_x = \mu_x(t) / (\mu_x(t) + \delta) = 0.5714 \]
\[ \overline{A}_y = \mu_y(t) / (\mu_y(t) + \delta) = 0.4 \]
\[ \overline{A}_{xy} = \mu_{xy}(t) / (\mu_{xy}(t) + \delta) = 0.6667 \]
\[ \overline{a}_{xy} = 1 / (\mu_{xy}(t) + \delta) = 5.556 \]

\[ \overline{A}_{xy} = \overline{A}_x + \overline{A}_y - \overline{A}_{xy} = 0.5714 + 0.4 - 0.6667 = 0.3047 \]

Premium = 0.304762/5.556 = 0.0549

Question #10
Key: B

\[ P_{40} = A_{40} / \ddot{a}_{40} = 0.16132 / 14.8166 = 0.0108878 \]
\[ P_{42} = A_{42} / \ddot{a}_{42} = 0.17636 / 14.5510 = 0.0121201 \]

\[ a_{45} = \ddot{a}_{45} - 1 = 13.1121 \]

\[ E\left[ \sum_{i=1}^{1000} K(42) \right] \geq 3 \] = 1000A_{45} - 1000P_{40} - 1000P_{42} a_{45}
\[ = 201.20 - 10.89 - (12.12)(13.1121) \]
\[ = 31.39 \]

Many similar formulas would work equally well. One possibility would be
1000 \(3V_{42} + (1000P_{42} - 1000P_{40})\), because prospectively after duration 3, this differs from the normal benefit reserve in that in the next year you collect 1000\(P_{40}\) instead of 1000\(P_{42}\).
For De Moivre’s Law:

\[ e_x = \frac{\omega - x}{2} \]
\[ i \dot{q}_x = \frac{1}{\omega - x} \]
\[ A_x = \sum_{k=b}^{\omega-x-1} v^{k+1} \dot{q}_x = \frac{1}{\omega - x} \sum_{k=b}^{\omega-x-1} v^{k+1} \]
\[ A_x = \frac{a_{\omega-x}}{\omega - x} \]
\[ \dot{a}_x = \frac{1 - A_x}{d} \]

\[ \dot{e}_{50} = 25 \Rightarrow \omega = 100 \text{ for typical annuitants} \]
\[ \dot{e}_y = 15 \Rightarrow y = \text{Assumed age} = 70 \]

\[ A_{70} = \frac{a_{30}}{30} = 0.45883 \]
\[ \dot{a}_{70} = 9.5607 \]
\[ 500000 = b \dot{a}_{20} \Rightarrow b = 52,297 \]
Question #12
Key: D

\[ p_x^{(r)} = p_x^{(l)} p_x^{(z)} = 0.8 \times 0.7 = 0.56 \]

\[ q_x^{(l)} = \frac{\ln(p_x^{(l)})}{\ln(p_x^{(r)})} q_x^{(r)} \text{ since UDD in double decrement table} \]

\[ = \frac{\ln(0.8)}{\ln(0.56)} 0.44 \]

\[ = 0.1693 \]

\[ 0.3 q_{x+0.1}^{(l)} = \frac{0.3 q_x^{(l)}}{1 - 0.1q_x^{(r)}} = 0.053 \]

To elaborate on the last step:

\[ 0.3 q_{x+0.1}^{(l)} = \frac{\text{Number dying from cause \[1\] between } x + 0.1 \text{ and } x + 0.4}{\text{Number alive at } x + 0.1} \]

Since UDD in double decrement,

\[ = \frac{l_x^{(r)} (0.3) q_x^{(l)}}{l_x^{(r)} (1 - 0.1q_x^{(r)})} \]
**Question #13**

Key: B

non absorbing matrix \( T = \begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.6 \end{pmatrix} \), the submatrix excluding “Terminated”, which is an absorbing state.

\[
I - T = \begin{pmatrix} 0.3 & -0.1 \\ -0.3 & 0.4 \end{pmatrix}
\]

\[
(I - T)^{-1} = \begin{pmatrix} 0.4 & 0.1 \\ 0.09 & 0.09 \\ 0.3 & 0.3 \\ 0.09 & 0.09 \end{pmatrix} = \begin{pmatrix} 4.44 & 1.11 \\ 3.33 & 3.33 \end{pmatrix}
\]

Future costs for a healthy = \( 4.44 \times 500 + 1.11 \times 3000 \)

= 5555

---

**Question #14**

Key: D

\[
T = \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0.3 & 0.6 & 0.1 \\ 0 & 0 & 1 \end{pmatrix} \quad T^2 = \begin{pmatrix} 0.52 & 0.13 & 0.35 \\ 0.39 & 0.39 & 0.22 \\ 0 & 0 & 1 \end{pmatrix}
\]

Actuarial present value (A.P.V.) prem = 800(1 + (0.7 + 0.1) + (0.52 + 0.13)) = 1,960

A.P.V. claim = 500(1 + 0.7 + 0.52) + 3000(0 + 0.1 + 0.13) = 1800

Difference = 160
Question #15
Key: A

Let $N_1, \ N_2$ denote the random variable for # of claims for Type I and II in 2 years
$X_1, \ X_2$ denote the claim amount for Type I and II
$S_1 = \text{total claim amount for type I in 2 years}$
$S_2 = \text{total claim amount for Type II at time in 2 years}$
$S = S_1 + S_2 = \text{total claim amount in 2 years}$

$\{S_1\} \rightarrow \text{compound poisson} \ \lambda_1 = 2 \times 6 = 12 \ \ X_1 \sim U(0,1)$
$\{S_2\} \rightarrow \text{compound poisson} \ \lambda_2 = 2 \times 2 = 4 \ \ X_2 \sim U(0, 5)$

$E(N_1) = Var(N_1) = 2 \times 6 = 12$
$E(S_1) = E(N_1)E(X_1) = (12)(0.5) = 6$
$Var(S_1) = E(N_1)Var(X_1) + Var(N_1)(E(X_1))^2$
$= (12)\frac{(1-0)}{12} + (12)(0.5)^2$
$= 4$

$E(N_2) = Var(N_2) = 2 \times 2 = 4$

With formulas corresponding to those for $S_1,$
$E(S_2) = 4 \times \frac{5}{2} = 10$
$Var(S_2) = 4 \times \frac{(5-0)^2}{12} + 4 \left(\frac{5}{2}\right)^2 = 33.3$
$E(S) = E(S_1) + E(S_2) = 6 + 10 = 16$

Since $S_1$ and $S_2$ are independent,
$Var(S) = Var(S_1) + Var(S_2) = 4 + 33.3 = 37.3$
$Pr(S > 18) = Pr \left( \frac{S - 16}{\sqrt{39.3}} > \frac{2}{\sqrt{37.3}} = 0.327 \right)$

Using normal approximation
$Pr(S > 18) = 1 - \Phi(0.327)$
$= 0.37$
Question #16
Key: D

Since the rate of depletion is constant there are only 2 ways the reservoir can be empty sometime within the next 10 days.

Way #1:  
There is no rainfall within the next 5 days

Way #2  
There is one rainfall in the next 5 days  
And it is a normal rainfall  
And there are no further rainfalls for the next five days

\[
\begin{align*}
\text{Prob (Way #1)} &= \text{Prob}(0 \text{ in 5 days}) = \exp(-0.2 \times 5) = 0.3679 \\
\text{Prob (Way #2)} &= \text{Prob}(1 \text{ in 5 days}) \times 0.8 \times \text{Prob}(0 \text{ in 5 days}) \\
&= 5 \times 0.2 \times \exp(-0.2 \times 5) \times 0.8 \times \exp(-0.2 \times 5) \\
&= 1 \times \exp(-1) \times 0.8 \times \exp(-1) = 0.1083
\end{align*}
\]

Hence Prob empty at some time = 0.3679 + 0.1083 = 0.476

Question #17
Key: C

Let \( X \) be the loss random variable,  
So \((X - 5)_+\) is the claim random variable.

\[
E(X) = \frac{10}{2.5 - 1} = 6.6
\]

\[
E(X \land 5) = \left(\frac{10}{2.5 - 1}\right) \left[1 - \left(\frac{10}{5+10}\right)^{2.5 - 1}\right] = 3.038
\]

\[
E(X - 5)_+ = E(X) - E(X \land 5) = 6.6 - 3.038 = 3.629
\]

Expected aggregate claims = \(E(N)E(X - 5)_+\)

\[
= (5)(3.629) = 18.15
\]
Question #18
Key: B

A Pareto ($\alpha = 2, \ \theta = 5$) distribution with 20% inflation becomes Pareto with $\alpha = 2, \ \theta = 5 \times 1.2 = 6$

In 2004, $E(X) = \frac{6}{2-1} = 6$

$$E(X \wedge 10) = \frac{6}{2-1} \left(1 - \left(\frac{6}{10+6}\right)^2\right) = 3.75$$

$$E(X-10)_+ = E(X) - E(X \wedge 10) = 6 - 3.75 = 2.25$$

$$\text{LER} = 1 - \frac{E(X-10)_+}{E(X)} = 1 - \frac{2.25}{6} = 0.625$$

Question #19
Key: A

Let $X =$ annual claims

$$E(X) = (0.75)(3) + (0.15)(5) + (0.1)(7) = 3.7$$

$$\pi = \text{Premium} = (3.7)\left(\frac{4}{3}\right) = 4.93$$

Change during year

$$= 4.93 - 3 = +1.93 \text{ with } p = 0.75$$

$$= 4.93 - 5 = -0.07 \text{ with } p = 0.15$$

$$= 4.93 - 7 = -2.07 \text{ with } p = 0.10$$

Since we start year 1 with surplus of 3, at end of year 1 we have 4.93, 2.93, or 0.93 (with associated probabilities 0.75, 0.15, 0.10).

We cannot drop more then 2.07 in year 2, so ruin occurs only if we are at 0.93 after 1 and have a drop of 2.07.

$$\text{Prob} = (0.1)(0.1) = 0.01$$
Question #20
Key: E

\[0.96 = e^{-(\mu_1 + \lambda)}\]
\[\mu_1 + \lambda = -\ln(0.96) = 0.04082\]
\[\mu_1 = 0.04082 - \lambda = 0.04082 - 0.01 = 0.03082\]

Similarly

\[\mu_2 = -\ln(0.97) - \lambda = 0.03046 - 0.01 = 0.02046\]
\[\mu_{xy} = \mu_1 + \mu_2 + \lambda = 0.03082 + 0.02046 + 0.01 = 0.06128\]
\[sP_{xy} = e^{-(5)(0.06128)} = e^{-0.3064} = 0.736\]
Question #21
Key: C

\[ A_{60} = 0.36913 \quad d = 0.05660 \]
\[ 2A_{60} = 0.17741 \]
and \( \sqrt{2A_{60} - A_{60}^2} = 0.202862 \)

Expected Loss on one policy is \( E[L(\pi)] = \left( 100,000 + \frac{\pi}{d} \right) A_{60} - \frac{\pi}{d} \)

Variance on one policy is \( \text{Var}[L(\pi)] = \left( 100,000 + \frac{\pi}{d} \right) \left( 2A_{60} - A_{60}^2 \right) \)

On the 10000 lives,
\[ E[S] = 10,000E[L(\pi)] \text{ and } \text{Var}[S] = 10,000 \text{Var}[L(\pi)] \]

The \( \pi \) is such that \( 0 - E[S]/\sqrt{\text{Var}[S]} = 2.326 \) since \( \Phi(2.326) = 0.99 \)

\[ \frac{10,000 \left( \frac{\pi}{d} - \left( 100,000 + \frac{\pi}{d} \right) A_{60} \right)}{100 \left( 100,000 + \frac{\pi}{d} \right) \sqrt{2A_{60} - A_{60}^2}} = 2.326 \]

\[ \frac{100 \left( \frac{\pi}{d} - \left( 100,000 + \frac{\pi}{d} \right) \right) (0.36913)}{\left( 100,000 + \frac{\pi}{d} \right) (0.202862)} = 2.326 \]

\[ \frac{0.63087\pi - 36913}{100,000 + \frac{\pi}{d}} = 0.004719 \]

\[ 0.63087\pi - 36913 = 471.9 = 0.004719 \frac{\pi}{d} \]

\[ \pi = \frac{36913 + 471.9}{0.63087 - 0.004719} = 59706 \]
\[ \pi = 59706 \times d = 3379 \]
\[ V = (0 + V + \pi)(1 + i) - (1000 + V - V) \times q_{75} \]
\[ = 1.05\pi - 1000q_{75} \]

Similarly,
\[ 2V = (1V + \pi) \times 1.05 - 1000q_{76} \]
\[ 3V = (2V + \pi) \times 1.05 - 1000q_{77} \]
\[ 1000 = 3V = (1.05^3 \pi + 1.05^2 \cdot \pi + 1.05\pi) - 1000 \times q_{75} \times 1.05^2 - 1000 \times 1.05 \times q_{76} - 1000 \times q_{77} \]
\[ \pi = \frac{1000 + 1000(1.05^2 q_{75} + 1.05q_{76} + q_{77})}{(1.05)^3 + (1.05)^2 + 1.05} \]
\[ = \frac{1000 \times (1 + 1.05^2 \times 0.05169 + 1.05 \times 0.05647 + 0.06168)}{3.310125} \]
\[ = \frac{1000 \times 1.17796}{3.310125} = 355.87 \]

* This equation is algebraic manipulation of the three equations in three unknowns \((1V, 2V, \pi)\).

One method – usually effective in problems where benefit = stated amount plus reserve, is to multiply the \(1V\) equation by 1.05, the \(2V\) equation by 1.05, and add those two to the \(3V\) equation: in the result, you can cancel out the \(1V\), and \(2V\) terms. Or you can substitute the \(1V\) equation into the \(2V\) equation, giving \(2V\) in terms of \(\pi\), and then substitute that into the \(3V\) equation.
Question #23  
Key: D

Actuarial present value (APV) of future benefits =  
\[= \frac{(0.005 \times 2000 + 0.04 \times 1000)}{1.06} + \frac{(1-0.005-0.04)(0.008 \times 2000 + 0.06 \times 1000)}{1.06^2}\]  
= 47.17 + 64.60  
= 111.77

APV of future premiums = \([1 + (1 - 0.005 - 0.04)/1.06]^{50}\)  
= (1.9009)(50)  
= 95.05  
\[E[1_1 | K(55) \geq 1] = 111.77 - 95.05 = 16.72\]
Question #24  
Key: D

\[
\hat{e}_0 = \hat{e}_{0.50} + 20 \hat{p}_0 \hat{e}_{20}
\]

\[
\hat{e}_0 = E[T] = \frac{50}{3-1} = 25
\]

\[
\hat{e}_{0.50} = E[T \wedge 20] = \frac{50}{3-1} \left(1 - \left(\frac{50}{50+20}\right)^{3-1}\right)
\]

\[
= 12.245
\]

\[
20 \hat{p}_0 = 1 - F_T(20) = 1 - \left(1 - \left(\frac{50}{50+20}\right)^3\right)
\]

\[
= 0.3644
\]

\[
25 = 12.245 + 0.3644 \hat{e}_{20}
\]

\[
\hat{e}_{20} = 35
\]

Alternate approach: if losses are Pareto with \( \theta = 50 \) and \( \alpha = 3 \), then claim payments per payment with an ordinary deductible of 20 are Pareto with \( \theta = 50 + 20 \) and \( \alpha = 3 \).

Thus \( E(T(20)) = \frac{50+20}{3-1} = 35 \)

This alternate approach was shown here for educational reasons: to reinforce the idea that many life contingent models and non-life models can have similar structure. We doubt many candidates would take that approach, especially since it involves specific properties of the Pareto distribution.
Question #25
Key: E

\(Q \geq P\) since in \(Q\) you only test at intervals; surplus below 0 might recover before the next test.
In \(P\), ruin occurs if you are ever below 0.
\(R \geq P\) since you are less likely to have surplus below 0 in the first \(N\) years (finite horizon) than forever.

Add the inequalities

\(Q + R \geq 2P\)

Also (why other choices are wrong)
\(S \geq Q\) by reasoning comparable to \(R \geq P\). Same testing frequency in \(S\) and \(Q\), but \(Q\) tests forever.
\(S \geq R\) by reasoning comparable to \(Q \geq P\). Same horizon in \(S\) and \(R\); \(R\) tests more frequently.
\(S \geq P\) \(P\) tests more frequently, and tests forever.

Question #26
Key: A

This is a nonhomogeneous Poisson process with intensity function

\[ \lambda(t) = 3 + 3t, \quad 0 \leq t \leq 2, \] where \(t\) is time after noon

Average \(\lambda = \int_1^2 \lambda(t) \, dt = \int_1^2 (3 + 3t) \, dt\)

\[ = \left[ 3t + \frac{3t^2}{2} \right]_1^2 \]

\[ = 7.5 \]

\[ f(2) = \frac{e^{-7.5} \cdot 7.5^2}{2!} = 0.0156 \]
Question #27  
Key: E  

\( X(t) - Y(t) \) is Brownian motion with initial value \(-2\) and \( \sigma^2 = 0.5 + 1 = 1.5 \)

By formula 10.6, the probability that \( X(t) - Y(t) \geq 0 \) at some time between 0 and 5 is

\[
2 \times \text{Prob}\left[ X(5) - Y(5) \geq 0 \right] \\
= 2 \times \left[ 1 - \Phi \left( \frac{2}{\sqrt{5(1.5)}} \right) \right] = 0.4652
\]

The 2 in the numerator of \( \frac{2}{\sqrt{5(1.5)}} \) comes from \( X(0) - Y(0) = -2 \); the process needs to move 2 to reach \( X(t) - Y(t) \geq 0 \).
Question #28
Key: B

\[ 2q_{80:84} = 2q_{80} + 2q_{84} - 2q_{80:84} \]

\[ = 0.5 \times 0.4 \times (1 - 0.6) + 0.2 \times 0.15 \times (1 - 0.1) \]
\[ = 0.10136 \]

Using new \( p_{82} \) value of 0.3

\[ 0.5 \times 0.4 \times (1 - 0.3) + 0.2 \times 0.15 \times (1 - 0.1) \]
\[ = 0.16118 \]

Change = 0.16118 − 0.10136 = 0.06

Alternatively,
\[ 2p_{80} = 0.5 \times 0.4 = 0.20 \]
\[ 3p_{80} = 2p_{80} \times 0.6 = 0.12 \]
\[ 2p_{84} = 0.20 \times 0.15 = 0.03 \]
\[ 3p_{84} = 2p_{84} \times 0.10 = 0.003 \]
\[ 2p_{80:84} = 2p_{80} + 2p_{84} - 2p_{80} \cdot 2p_{84} \text{ since independent} \]
\[ = 0.20 + 0.03 - (0.20)(0.03) = 0.224 \]
\[ 3p_{80:84} = 3p_{80} + 3p_{84} - 3p_{80} \cdot 3p_{84} \]
\[ = 0.12 + 0.003 - (0.12)(0.003) = 0.12264 \]
\[ 2q_{80:84} = 2p_{80:84} - 3p_{80:84} \]
\[ = 0.224 - 0.12264 = 0.10136 \]

Revised
\[ 3p_{80} = 0.20 \times 0.30 = 0.06 \]
\[ 3p_{80:84} = 0.06 + 0.003 - (0.06)(0.003) \]
\[ = 0.06282 \]
\[ 2q_{80:84} = 0.224 - 0.06282 = 0.16118 \]

change = 0.16118 − 0.10136 = 0.06
Question #29
Key: B

\[ e_x = p_x + p_x e_{x+1} \Rightarrow p_x = \frac{e_x}{1 + e_{x+1}} = \frac{8.83}{9.29} = 0.95048 \]

\[ \dot{a}_x = 1 + v p_x + v^2 p_x + \ldots \]

\[ \ddot{a}_{x:2} = 1 + v + v^2 p_x + \ldots \]

\[ \ddot{a}_{x:2} - \ddot{a}_x = v q_x = 5.6459 - 5.60 = 0.0459 \]

\[ v(1 - 0.95048) = 0.0459 \]

\[ v = 0.9269 \]

\[ i = \frac{1}{v} - 1 = 0.0789 \]
Let \( \pi \) be the benefit premium
Let \( kV \) denote the benefit reserve a the end of year \( k \).

For any \( n, (nV + \pi)(1+i) = (q_{25+n} \times n+tV + p_{25+n} \times n+tV) \)
\[ = n+tV \]
Thus \( V = (0V + \pi)(1+i) \)

\[ _2V = (1V + \pi)(1+i) = (\pi(1+i) + \pi)(1+i) = \pi \ddot{s}_{2|} \]
\[ _3V = (2V + \pi)(1+i) = (\pi \ddot{s}_{3|} + \pi)(1+i) = \pi \ddot{s}_{3|} \]

By induction (proof omitted)
\[ nV = \pi \ddot{s}_{n|} \]

For \( n = 35, \ nV = \ddot{a}_{60} \) (actuarial present value of future benefits; there are no future premiums)
\[ \ddot{a}_{60} = \pi \ddot{s}_{55|} \]
\[ \pi = \ddot{a}_{60} \ddot{s}_{55|} \]

For \( n = 20, \ _{20}V = \pi \ddot{s}_{20|} \)
\[ = \left( \frac{\ddot{a}_{60}}{\ddot{s}_{55|}} \right) \ddot{s}_{20|} \]

Alternatively, as above
\( (nV + \pi)(1+i) = n+tV \)

Write those equations, for \( n = 0 \) to \( n = 34 \)
\[ 0: (0V + \pi)(1+i) = V \]
\[ 1: (1V + \pi)(1+i) = _1V \]
\[ 2: (2V + \pi)(1+i) = _2V \]
\[ \vdots \]
\[ 34: (_{34}V + \pi)(1+i) = _{35}V \]

Multiply equation \( k \) by \( (1+i)^{34-k} \) and sum the results:
\[ (0V + \pi)(1+i)^{35} + (1V + \pi)(1+i)^{34} + (2V + \pi)(1+i)^{33} + \cdots + (_{34}V + \pi)(1+i) = \]
\[ _1V(1+i)^{34} + _2V(1+i)^{33} + _3V(1+i)^{32} + \cdots + _{34}V(1+i) + _{35}V \]
For $k = 1, 2, \cdots, 34$, the $kV (1+i)^{35-k}$ terms in both sides cancel, leaving

$$0V (1+i)^{35} + \pi \left[ (1+i)^{35} + (1+i)^{34} + \cdots + (1+i) \right] = 35V$$

Since $0V = 0$

$$\pi \dddot{s}_{35|} = 35V$$

$$= \dddot{a}_{60}$$

(see above for remainder of solution)

This technique, for situations where the death benefit is a specified amount (here, 0) plus the benefit reserve is discussed in section 8.3 of Bowers. This specific problem is Example 8.3.1.
Question #31
Key: B

\[ \mu_y(t) = \frac{t q_x t p_x \mu(x + t) + t q_x t p_y \mu(y + t)}{t q_x x t p_x + t p_x x t q_y + t p_x x t p_y} \]

For \((x) = (y) = (50)\)

\[ \mu_{50:50}(10.5) = \frac{(10.5 q_{50})(10 p_{50}) q_{60} \cdot 2}{(10.5 q_{50})(10.5 p_{50}) \cdot 2 + (10.5 p_{50})^2} = \frac{(0.09152)(0.91478)(0.01376)(2)}{(0.09152)(0.90848)(2) + (0.90848)^2} = 0.0023 \]

where

\[ 10.5 p_{50} = \frac{\frac{1}{2}(l_{60} + l_{61})}{l_{50}} = \frac{\frac{1}{2}(8,188,074 + 8,075,403)}{8,950,901} = 0.90848 \]

\[ 10.5 q_{50} = 1 - 10.5 p_{50} = 0.09152 \]

\[ 10 p_{50} = \frac{8,188,074}{8,950,901} = 0.91478 \]

\[ 10.5 p_{50} \mu(50 + 10.5) = (10 p_{50}) q_{60} \text{ since UDD} \]

Alternatively, \((10 + t) p_{50} = 10 p_{50} + t p_{60}\)

\[ (10 + t) p_{50:50} = (10 p_{50})^2 + (t p_{60})^2 \]

\[ (10 + t) p_{50:50} = 2(10 p_{50} t p_{60} - (10 p_{50})^2 (t p_{60})^2 \]

\[ = 2(10 p_{50})(1 - tq_{60}) - (10 p_{50})^2 (1 - tq_{60})^2 \text{ since UDD} \]

Derivative \(-2(10.5 p_{50})q_{60} + 2(10 p_{50})^2 (1 - tq_{60})q_{60} \)

Derivative at \(10 + t = 10.5\) is

\[ -2(0.91478)(0.01376) + (0.91478)^2 (1 - (0.5)(0.01376))(0.01376) = -0.0023 \]

\[ 10.5 p_{50:50} = 2(0.90848) - (0.90848)^2 \]

\[ = 0.99162 \]

\[ \mu \text{ (for any sort of lifetime)} = \frac{-dp}{dt} = \frac{-(-0.0023)}{0.99162} = 0.0023 \]
Question #32
Key: E

\[ E(W) = \frac{1}{4} \int \sum_{i=0}^{\infty} 2^i \Pr(N = i| \lambda) d\lambda \quad \left[ \frac{1}{4} \right. \text{ is the density of } \lambda \text{ on } [0, 4]. \]

\[ = \frac{1}{4} \int_{0}^{4} P(2| \lambda) d\lambda \quad \text{[see note]} \]

\[ = \frac{1}{4} \int_{0}^{4} e^{\lambda(2-1)} d\lambda \quad \text{[using formula from tables for the pgf of the Poisson]} \]

\[ = \frac{1}{4} e^{\lambda} \bigg|_{0}^{4} = \frac{1}{4}(e^{4} - 1) \]

\[ = 13.4 \]

Note: the probability generating function (pgf) is \( P(Z) = \sum_{k=0}^{\infty} p_k Z^k \) so the integrand is \( P(2) \), or in this case \( P(2| \lambda) \) since \( \lambda \) is not known.

Alternatively,

\[ E(W) = \frac{1}{4} \int \sum_{i=0}^{\infty} 2^i \Pr(N = i| \lambda) d\lambda \]

\[ = \frac{1}{4} \int_{0}^{4} \sum_{i=0}^{\infty} \frac{2^i e^{-\lambda} \lambda^i}{i!} d\lambda \]

\[ = \frac{1}{4} \int_{0}^{4} \sum_{i=0}^{\infty} \frac{e^{-\lambda} (2\lambda)^i}{i!} d\lambda \]

We know \( \sum_{i=0}^{\infty} \frac{e^{-2\lambda} (2\lambda)^i}{i!} = 1 \) since \( \frac{e^{-2\lambda} (2\lambda)^i}{i!} \) is \( f(i) \) for a Poisson with mean \( Z\lambda \)

so \( \sum_{i=0}^{\infty} \frac{e^{-\lambda} (2\lambda)^i}{i!} = \frac{e^{-\lambda}}{e^{-2\lambda}} = e^{\lambda} \)

Thus \( E(W) = \frac{1}{4} \int_{0}^{4} e^{\lambda} d\lambda \)

\[ = \frac{1}{4} e^{\lambda} \bigg|_{0}^{4} = \frac{1}{4}(e^{4} - 1) \]

\[ = 13.4 \]
Question #33

Key: A or E

\[ E(S) = \lambda E[X] = 2/3(1/4 + 2/4 + 3/2) = 2/3 \times 9/4 = 3/2 \]

\[ \text{Var}(S) = \lambda E[X^2] = 2/3(1/4 + 4/4 + 9/2) = 23/6 \]

So cumulative premium to time 2 is \( 2\left(\frac{3}{2} + 1.8\sqrt{23/6}\right) = 10 \), where the expression in parentheses is the annual premium.

Times between claims are determined by \(-1/\lambda \log u\) and are 0.43, 0.77, 1.37, 2.41
So 2 claims before time 2 (second claim is at 1.20; third is at 2.57)

Sizes are 2, 3, 1, 3, where only the first two matter.

So gain to the insurer is 10-(2+3) = 5

Note: since the problem did not specify that we wanted the gain or loss from the insurer’s viewpoint, we gave credit to answer A; a loss of 5 from the insured’s viewpoint.
To get number of claims, set up cdf for Poisson:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$F(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.135</td>
<td>0.135</td>
</tr>
<tr>
<td>1</td>
<td>0.271</td>
<td>0.406</td>
</tr>
<tr>
<td>2</td>
<td>0.271</td>
<td>0.677</td>
</tr>
<tr>
<td>3</td>
<td>0.180</td>
<td>0.857</td>
</tr>
</tbody>
</table>

0.80 simulates 3 claims.

\[ F(x) = 1 - \left( \frac{500}{x+500} \right)^2 = u, \quad \text{so} \ x = \left( 1 - u \right)^{-1/2} 500 - 500 \]

0.6 simulates 290.57
0.25 simulates 77.35
0.7 simulates 412.87

So total losses equals 780.79

Insurer pays \((0.80)(750) + (780.79 - 750) = 631\)
Question #35
Key: D

\[
\mu_{x}^{(t)}(t) = \mu_{x}^{(1)}(t) + \mu_{x}^{(2)}(t) = 0.01 + 2.29 = 2.30
\]

\[
P = P \int_{0}^{2} e^{-0.1t} e^{-2.3t} \times 2.29 dt + 50,000 \int_{0}^{2} e^{-0.1t} e^{-2.3t} \times 0.01 dt + 50,000 \int_{2}^{\infty} e^{-0.1t} e^{-2.3t} \times 2.3 dt
\]

\[
P = P \left[ 1 - 2.29 \times \frac{1 - e^{-2(2.4)}}{2.4} \right] = 50000 \left[ 0.01 \times \frac{1 - e^{-2(2.4)}}{2.4} + 2.3 \times \frac{e^{-2(2.4)}}{2.4} \right]
\]

\[
P = 11,194
\]

Question #36
Key: D

\[
\mu^{(accid)} = 0.001
\]

\[
\mu^{(total)} = 0.01
\]

\[
\mu^{(other)} = 0.01 - 0.001 = 0.009
\]

Actuarial present value = \[\int_{0}^{\infty} 500,000 e^{-0.05t} e^{-0.01t} (0.009) dt \]

\[+10 \int_{0}^{\infty} 50,000 e^{0.04t} e^{-0.05t} e^{-0.01t} (0.001) dt \]

\[= 500,000 \left[ \frac{0.009}{0.06} + \frac{0.001}{0.02} \right] = 100,000 \]
Question #37
Key: B

Variance = \( \nu^{15} p_x \nu^{15} q_x \)  
Expected value = \( \nu^{15} p_x \)

\[
\begin{align*}
\nu^{15} p_x \nu^{15} q_x &= 0.065 \\
\nu^{15} q_x &= 0.065 \Rightarrow \nu^{15} q_x &= 0.3157
\end{align*}
\]

Since \( \mu \) is constant

\[
\nu^{15} q_x = \left(1 - (p_x)^{15}\right)
\]

\[
(p_x)^{15} = 0.6843
\]

\[
p_x = 0.975
\]

\[
q_x = 0.025
\]

Question #38
Key: E

\[
(1) \quad \nu^{15} A = \left(10 \nu^{15} A + 0\right) \frac{(1+i)}{p_{x+10}} - \frac{q_{x+10}}{p_{x+10}} \times 1000
\]

\[
(2) \quad \nu^{15} B = \left(10 \nu^{15} B + \pi^B\right) \frac{(1+i)}{p_{x+10}} - \frac{q_{x+10}}{p_{x+10}} \times 1000
\]

\[
(1)-(2) \quad \nu^{15} A - \nu^{15} B = \left(10 \nu^{15} A - 10 \nu^{15} B - \pi^B\right) \frac{(1+i)}{p_{x+10}}
\]

\[
= (101.35 - 8.36) \frac{(1.06)}{1-0.004}
\]

\[
= 98.97
\]
Question #39
Key: A

Actuarial present value Benefits = \[
\frac{(0.8)(0.1)(10,000)}{1.06^2} + \frac{(0.8)(0.9)(0.097)(9,000)}{1.06^3}
\]
\[= 1,239.75\]

\[
1,239.75 = P\left(1 + \frac{(0.8)}{1.06} + \frac{(0.8)(0.9)}{1.06^2}\right)
\]
\[= P(2.3955)\]
\[P = 517.53 \implies 518\]

Question #40
Key: C

<table>
<thead>
<tr>
<th>Event</th>
<th>Prob</th>
<th>Present Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x = 0)</td>
<td>((0.05))</td>
<td>15</td>
</tr>
<tr>
<td>(x = 1)</td>
<td>((0.95)(0.10) = 0.095)</td>
<td>(15 + 20/1.06 = 33.87)</td>
</tr>
<tr>
<td>(x \geq 2)</td>
<td>((0.95)(0.90) = 0.855)</td>
<td>(15 + 20/1.06 + 25/1.06^2 = 56.12)</td>
</tr>
</tbody>
</table>

\[E[X] = (0.05)(15) + (0.095)(33.87) + (0.855)(56.12) = 51.95\]

\[E[X^2] = (0.05)(15)^2 + (0.095)(33.87)^2 + (0.855)(56.12)^2 = 2813.01\]

\[Var[X] = E[X^2] - E[X]^2 = 2813.01 - (51.95)^2 = 114.2\]