Polychromatic Colorings of the Integers

We show that if $S$ is any set of 4 integers then there is a 3-coloring of $\mathbb{Z}$ such that every additive translate of $S$ gets all 3 colors. This proves a conjecture of Newman that the codensity of any set of 4 integers in $\mathbb{Z}$ is at most $1/3$ (the codensity of a finite set $S$ of integers is the minimum density of a set $T$ in $\mathbb{Z}$ such that $S + T = \mathbb{Z}$).

Loosely speaking, if $L$ is any large structure consisting of some elements, and $\mathcal{F}$ is a family of substructures of $L$, we say a $k$-coloring of the elements of $L$ is $\mathcal{F}$-polychromatic if every substructure in $\mathcal{F}$ gets all $k$ colors. The polychromatic number of $\mathcal{F}$ in $L$ is the largest $k$ such that there exists an $\mathcal{F}$-polychromatic $k$-coloring. We will talk briefly about polychromatic colorings in other settings besides the integers and suggest some questions for further thought.