Non-negative Integer Solutions of Linear Equations and their Simplicial Complexes

Let $A$ be an $n \times m$-matrix of rank $n$ with integer entries. Let $S$ denote the set of all solutions $x$ to the equation $Ax = 0$, where $x$ is a vector with nonnegative integer entries. The Hilbert basis of $S$ is the minimal subset $H$ of $S$ with the property that any solution $x$ in $S$ can be written as a nonnegative integer combination of solutions in $H$. For $n = 1$, we recently gave a geometric characterization of the Hilbert basis $H$. In our quest to extend this characterization to $n > 1$, we introduce simplicial complexes that can be associated with the set of nonnegative solutions $S$. We then prove that when $n = 2$, the resulting simplicial complex is always “regular” in a sense that generalizes the traditional notion of regularity in graphs.