Undergraduate Colloquium

Title: Combinatorial Properties of the Stern Sequence

Speaker: Professor Bruce Reznick
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Location: STV 121
Time: 3:00 - 4:00 pm on Thursday (04/04/2019)

Abstract: In this talk, we will survey some old and new results about the Stern sequence, a highly underappreciated mathematical object. It is defined by the recurrence

$$s(0) = 0, s(1) = 1, s(2n) = s(n), s(2n+1) = s(n) + s(n+1).$$

It is most easily written by imagining a Pascal triangle with memory, and starting with (1,1). The rows of the resulting "diatomic" array give s(n) for $2^r \le n \le 2^{r+1}$:

(r=0): 1 1
(r=1): 1 2 1
(r=2): 1 3 2 3 1
(r=3): 1 4 3 5 2 5 3 4 1

Stern himself proved in 1858 that every pair of relatively prime positive integers (a,b) occurs exactly once as the pair (s(n),s(n+1)), and the binary representation of n is encoded by the continued fraction representation of a/b. The maximum values in the r-th row is the (r+2)-nd Fibonacci number. The sum of the entries in the r-th row is $3^r + 1$; the sum of the cubes of the entries is $9*7^{r-1}$ for $r > 0$. The Stern sequence also has many interesting divisibility properties: s(n) is even iff n is a multiple of 3; the set of n for which s(n) is a multiple of 3 has a simple recursive description. The set of n for which s(n) is a multiple of the prime p has density 1/(p+1). Further, s(n) counts the number of binary representations of n-1, if one allows digits from {0,1,2}. The Stern sequence affords a clear understanding of the alluringly-named Minkowski ?-function, which gives a strictly increasing map from [0,1] to itself, taking the rationals to the dyadic rationals, and the quadratic irrationals to the non-dyadic rationals. This talk will be elementary and is intended to be accessible to alert undergraduates.