Topic: Properties of Exponents

Definition:

Exponents represent repeated multiplication. The expression 2^3 represents $2 \cdot 2 \cdot 2$, which we can calculate to be 8. The expression x^4 represents $x \cdot x \cdot x \cdot x$. We cannot assign a value to the expression without knowing the value of x, so the answer is best left as x^4.

A special case is when a number is raised to the first power. For any number x, $x^1 = x$. Raising a number to the first power gives back the same number, similar to multiplying by 1.

Another special case is when a number is raised to the zero power. For any number x, $x^0 = 1$. Any number raised to the 0 power gives back 1 (except when $x = 0$, but this is a minor technicality). When we multiply zero x's, we get 1 (whereas when we add zero x's, we get 0).

Examples:

1. Find the value of 3^4.

We start by multiplying two 3's, to get $3 \cdot 3 = 9$. Then we multiply by a third 3, to get $9 \cdot 3 = 27$. Finally we multiply by a fourth 3, to get $27 \cdot 3 = 81$.

2. Find the value of $2^3 5^2$.

Since there is no symbol between the 2^3 and the 5^2, the implied operation is multiplication. We first calculate $2^3 = 8$ and $5^2 = 25$. Then we multiply these results to get $2^3 5^2 = 8 \cdot 25 = 200$.

Negative exponents:

A negative exponent corresponds to taking the reciprocal. For example, $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$. The easiest way to handle negative exponents is to put it in the denominator and drop the negative sign; or, if it is already in the denominator, move it to the numerator.

Caution: Taking a positive number to a negative exponent does not result in a negative number.

Examples:

1. Express $x^{-3} y^4 z^{-2}$ without using negative exponents.

Because they have negative exponents, the x^{-3} and the z^{-2} should be moved to the denominator, with the negatives removed. So we get $\frac{y^4}{x^3 z^2}$.

2. Express $\frac{x^{-2}}{y^{-3} z^5}$ without using negative exponents.

The x should move down, the y should move up, and the z should not move. So we get $\frac{y^3}{x^2 z^5}$.

3. Find the value of 3^{-4}.

Because of the negative exponent, we move the expression to the denominator and drop the negative sign: $3^{-4} = \frac{1}{3^4} = \frac{1}{81}$.

Addition property of exponents

When the same base is raised to two exponents and the results are multiplied, we can combine the result into one exponent by adding the exponents. Formally, the property is: $\boxed{x^a x^b = x^{a+b}}$

As justification: $x^a x^b = \overbrace{x \cdot x \cdot \cdots \cdot x}^{a} \cdot \overbrace{x \cdot x \cdot \cdots \cdot x}^{b} = \overbrace{x \cdot x \cdot \cdots \cdot x}^{a+b} = x^{a+b}$

Caution: This property can only be applied when the bases of the two factors are the same.

Examples

1. Find the value of $2^4 2^{-6} 2^3$.

Since the base of each exponential is 2, we can apply the addition property. $2^4 2^{-6} 2^3$ becomes $2^{4-6+3} = 2^1 = 2$. This is far easier than finding the value of each factor and then multiplying.

2. Write $x^{-3} x^8 x^9$ using a single exponent.

Since the base of each exponential is x, we can apply the addition property. $x^{-3} x^8 x^9$ becomes $x^{-3+8+9} = x^{14}$.

Subtraction property of exponents

When the same base is raised to two exponents and the results are divided, we can combine the result into one exponent by subtracting the exponents. Formally, the property is: $\boxed{\frac{x^a}{x^b} = x^{a-b}}$

Multiplication property of exponents

When a base is raised to a power, and the result is then raised to another power, the exponents are multiplied. Formally, the property is: $\boxed{(x^a)^b = x^{a \cdot b}}$

Distribution of exponents

When a product of two (or more) numbers is raised to a power, the exponent distributes onto each factor of the product. Formally, the property is: $\boxed{(xy)^a = x^a y^a}$

Radicals

The square root of a number x is defined as the number which, when squared, gives x. If x is positive, there will be two such numbers; the radical symbol $\sqrt{x}$ always refers to the positive one. For example, $\sqrt{4}$ is 2 and not -2.

The nth root of x is the number which, when raised to the nth power, gives x. It is denoted by $\sqrt[n]{x}$. For example, $\sqrt[3]{8} = 2$ because $2^3 = 8$.

Fractional exponents

Raising a number to the $1/n$ power is the same as taking the nth root. Raising a number to the m/n power is the same as taking the nth root of the mth power of the number. For example, $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$.

Note that $\sqrt[4]{x}$ and $x^{1/4}$ represent the same thing. Using fractional exponents generally makes it easier to the properties given above.