Connecting Derivatives and Antiderivatives

For \( y = f(x) \), is there a function \( F \) such that \( F' = f \)? We say that \( F \) is the antiderivative of \( f \) and we call \( f \) the derivative of \( F \).

\[
\begin{align*}
  f(x) & = \frac{1}{x} \Rightarrow F(x) = \ln(x) + C \\
  f(x) & = \cos(x) \Rightarrow F(x) = \sin(x) + C \\
  f(x) & = e^{x} \Rightarrow F(x) = e^{x} + C \\
  f(x) & = e^{\sin(x)} \Rightarrow F(x) = ?
\end{align*}
\]

In these first four cases, \( f \) and \( F \) are related as derivatives and antiderivatives, because \( F' = f \).

Local Linearity

Because a derivative describes how a function is changing at a single point, it is used to describe the slope of a line tangent to a curve at a given point.

Example: Shown above is the graph of \( f(x) = 2x^2 - 1 \) and a tangent line drawn to the curve at the point (1,1).

- What is the slope of that tangent line? The value of the derivative at \( x = 1 \) reveals the slope: \( f'(x) = 4x \Rightarrow f'(1) = 4 \).
- We can then use that slope, together with the ordered pair representing the point of tangency, to write an equation for the tangent line:
  \[
  y - 1 = 4(x - 1) \Rightarrow y = 4x - 3.
  \]
Try graphing a function together with its tangent line at some point on the graph of the original function. Repeatedly zoom in on that point of tangency with both the original function and the tangent line being re-graphed. What do you observe?

Writing linear representations for nonlinear functions—as models of them or as approximations for them—is a common and powerful application of derivatives. For many non-linear functions, the function’s behavior very close to a given point can be well approximated with a linear relationship.

**Position, Velocity, and Acceleration**

A common application in calculus focuses on the relationships among the position, velocity, and acceleration of some object. If we know the position of the object according to some function rule—revealed through a symbolic representation or a table of values or a graphical representation—then the derivative of the position function describes to us the instantaneous rate of change of that position. That rate of change is called velocity. Likewise, the rate of change of velocity—the derivative of the derivative of position—gives us acceleration, a description of how the velocity is changing.

**Position** describes where an object is, **velocity** describes how its position is changing, and **acceleration** describes how the velocity is changing.

Example: Consider the position function $s(t) = 3t^3 - 8t + 4$ for an object moving along a number line for values of $t$ within the interval $[-5, 5]$. From the position function we can derive velocity and acceleration:

- velocity: $v(t) = s'(t) = 9t^2 - 8$
- acceleration: $a(t) = v'(t) = s''(t) = 18t$

The graph shows all three function plots.

The three functions, in turn, can be used to answer a variety of questions about the context. Questions such as the following require application of many fundamental characteristics of derivatives and antiderivatives.

- **When is the object moving to the right?** The object moves to the right when its velocity is positive. As an approximation, plot (ii) above shows that the velocity is positive for values of $t$ less than $-1$ and for values of $t$ greater than $1$. For greater accuracy, set $v(t) = 0$, solve for $t$, and then choose the appropriate intervals. This reveals that the object moves to the right when $-5 \leq t < -\frac{2\sqrt{2}}{3}$ and when $\frac{2\sqrt{2}}{3} < t \leq 5$, where $\frac{2\sqrt{2}}{3} \approx 0.9428$.

Note that we have not included the points $\pm \frac{2\sqrt{2}}{3}$ in their respective intervals. Why not?
• When is the object’s velocity decreasing? When a function is decreasing, its derivative is negative. The derivative of velocity is acceleration, so we determine where acceleration is negative. This is true for \(-5 \leq t < 0\).

• How far has the object traveled on the interval \(0 \leq t \leq 2\)? Here, we mean the total distance the object has moved during the time span, not simply its net change in position. From the graph, we see that the object changed direction once on the interval \(0 \leq t \leq 2\) (the velocity changed from negative to positive at \(t = \frac{2\sqrt{2}}{3}\)), so we need to determine the absolute change in position on \(0 \leq t \leq \frac{2\sqrt{2}}{3}\) and on \(\frac{2\sqrt{2}}{3} \leq t \leq 2\) and then add the results:

\[
s(0) - s\left(\frac{2\sqrt{2}}{3}\right) + \left| s\left(\frac{2\sqrt{2}}{3}\right) - s(2) \right| = -16 \cdot \frac{2^\frac{3}{2}}{9} + \frac{32\sqrt{2}}{3} + 8 \approx 18.0566
\]

• What is the object’s displacement on the interval \(0 \leq t \leq 2\)? Here, we want the net change in position over the time span, so we compare \(s(0) = 4\) and \(s(2) = 12\). The object has moved 8 units to the right.

• When is the object’s velocity at its least? The minimum velocity occurs when the velocity function reaches an absolute minimum on the given interval. From the graph, we see this occurs when \(t = 0\). We can also determine this by solving \(a(t) = 0\) and checking those results to see whether the solutions lead to extreme values.

• What is the average velocity of the object on the interval \(-2 \leq t \leq 2\)? We are looking for how the object’s position has changed over the time span. We have \(\frac{s(2) - s(-2)}{2 - (-2)} = \frac{12 - (-4)}{4} = \frac{16}{4} = 4\), so the average velocity is 4 units per second. What is the graphical interpretation of this calculation?

Applications of Derivatives and Definite Integrals: Rates of Change and Accumulations

In general, derivatives describe instantaneous rate of change: At a particular point in time or for some particular input value, the derivative describes how the function is changing. Definite integrals describe accumulation: Over a span of time or for a range of input values, the definite integral describes total change.

Example: Suppose that for any time \(t\), in hours, where \(t\) ranges from 0 to 12 and 0 represents midnight, rain is falling at a rate of \(\frac{1}{36} t^2\) inches per hour. This is a derivative, for it describes the rate of change in the rainfall for any instant in time within the given time span. When \(t = 2\) hours (i.e., 2 o’clock in the morning), rain is falling at the rate of \(\frac{1}{36} \cdot 2^2 = \frac{1}{9}\) inch per hour.
To determine how much rain has fallen in that 12-hour time span, we apply a definite integral, for we want to know the total change in the amount of rain that has fallen over a time span (accumulation): \[ \int_{0}^{12} \frac{1}{36} t^2 \, dt = 16 \text{ inches of rain in all.} \]

**Application of the Definite Integral: Areas Under a Curve**

Through calculation and summation of the areas of rectangles, together with application of the limit concept, we can show that a definite integral represents the area under a curve for a specified x-axis interval. In fact, we often introduce and motivate integration through this context.

**Example:** Determine the area under the curve \( y = x^2 \) for the interval \( 1 \leq x \leq 3 \). Applying a definite integral, we have \[ \int_{1}^{3} x^2 \, dx = \frac{26}{3} \]. The plot here shows a shaded region as the area in question.