When the integration process is not immediately obvious, it may be possible to ‘reduce’ the integral to a well-known form by means of substitution.”

In any attempt to evaluate an integral, we try to determine what function has the integrand as its derivative. That is, for $\int f(x) \, dx$, we seek a function $F$ such that $F' = f$. In many cases we can identify $F$ by inspection, because we know a lot about functions and their derivatives.

We can look at $\int x^2 \, dx$ and quite readily determine that $\int x^2 \, dx = \frac{1}{3} x^3 + C$, because for $f(x) = x^2$, the function $F(x) = \frac{1}{3} x^3 + C$ satisfies the requirement that $F' = f$. Likewise, there is no calculation or symbol manipulation required to know or determine that $\int \cos(x) \, dx = \sin(x) + C$, because we know the derivative of $\sin(x)$ is $\cos(x)$.

Just as there are many cases for which we can readily identify an integral, there are lots more integrals for which it is difficult to simply identify by inspection the function that gives the integrand as its derivative. Hence the need for methods of integration.

The method of substitution is widely used with a variety of integrals, but all such integrals have one thing in common: Within the integrand appear both a function and (some multiple of) its derivative. Consider the following examples.
For each of these integrals, and countless others, we can make what is traditionally called a *u-substitution* that involves a function and its derivative, both appearing within the integrand.

**Example 1:** Determine \( \int 2x(x^2-1)^4 \, dx \) by making the substitution \( u = x^2 - 1 \).

From the table above, we know that \( \frac{du}{dx} = 2x \).

Although the symbol \( \frac{du}{dx} \) is just a way to represent a derivative, we now treat \( du \) and \( dx \) as if each were a quantity that could be manipulated in and of itself:

\[
du = 2x \, dx
\]

If we rewrite the integral in an equivalent form, \( \int (x^2-1)^4 \, 2x \, dx \), we can now make the following substitutions to express the integral in terms of the variable \( u \).

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1. From the standpoint of determining derivatives, we might informally think about \( du \) and \( dx \) as small changes in the output and input variables, something similar to what we mean by \( \Delta u \) and \( \Delta x \) when calculating the slope of a line.
Let \( u = x^2 - 1 \)
and therefore \( du = 2x \, dx \).

This results in

\[
\int 2x(x^2 - 1)^4 \, dx = \int (x^2 - 1)^4 \, 2x \, dx = \int u^4 \, du
\]

The last integral is one we can now determine by inspection, for

\[
\int u^4 \, du = \frac{1}{5} u^5 + C.
\]

The final step is to rewrite the value of the integral in terms of the original variable \( x \). We do this by substituting \( u = x^2 - 1 \) into the value of the integral:

\[
\frac{1}{5} u^5 + C = \frac{1}{5} (x^2 - 1)^5 + C.
\]

We have now come full circle and can express our solution to the original integral:

\[
\int 2x(x^2 - 1)^4 \, dx = \frac{1}{5} (x^2 - 1)^5 + C.
\]

**Example 2:** Solve \( \int \sin(2x) \cos(2x) \, dx \) using a \( u \)-substitution.

Reviewing the information in the table above, we know that a function and its derivative appear within this integral expression. We let \( u = \sin(2x) \) and proceed as in Example 1. We have

\[
\frac{du}{dx} = 2 \cos(2x) \Rightarrow du = 2 \cos(2x) \, dx
\]

Notice that the last expression is not quite identical to the derivative as it appears in the original integral. We carry out one step of symbol manipulation to make them identical:

\[
du = 2 \cos(2x) \, dx \Rightarrow \frac{1}{2} \, du = \cos(2x) \, dx.
\]

We now can make the substitutions necessary to create an integral we can solve by inspection:

Let \( u = \sin(2x) \)
and therefore \( \frac{1}{2} \, du = \cos(2x) \, dx \).

We now have

\[
\int \sin(2x) \cos(2x) \, dx = \int u \, \frac{1}{2} \, du = \frac{1}{2} \int u \, du.
\]
The last integral we can solve by inspection:
\[
\frac{1}{2} \int u \, du = \frac{1}{2} \left( \frac{u^2}{2} + C \right) = \frac{1}{4} u^2 + C.
\]
We complete the process by expressing the solution in terms of \( x \):
\[
\frac{1}{4} u^2 + C = \frac{1}{4} [\sin(2x)]^2 + C.
\]
This shows that
\[
\int \sin(2x)\cos(2x) \, dx = \frac{1}{4} [\sin(2x)]^2 + C.
\]

**Example 3:** Evaluate \( \int x e^{x^2} \, dx \).

Let \( u = x^2 \). Then \( \frac{du}{dx} = 2x \Rightarrow du = 2x \, dx \) or \( \frac{1}{2} du = x \, dx \).

Substituting, \( \int x e^{x^2} \, dx = \int e^u \cdot \frac{1}{2} \, du = \frac{1}{2} \int e^u \, du \),

and \( \frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C \).

Therefore, \( \int x e^{x^2} \, dx = \frac{1}{2} e^{x^2} + C \).

The method of substitution reverses a derivative process we learned that we call the chain rule. Recall that, using the chain rule,
\[
\frac{d}{dx} \left[ f(g(x)) \right] = f'(g(x)) \cdot g'(x)
\]
To evaluate \( \int f(g(x))g'(x) \, dx \), we let \( u = g(x) \) and therefore
\[
\frac{du}{dx} = g'(x) \Rightarrow du = g'(x) \, dx. \text{ So } \int f(g(x))g'(x) \, dx = \int f(u) \, du = F(u) + C,
\]
where \( F \) is the antiderivative of \( f \).

**Example 4:** Determine the value of \( \int \frac{x + 2}{x^2 + 4x + 7} \, dx \).

Let \( u = x^2 + 4x + 7 \).

Then \( \frac{du}{dx} = 2x + 4 \Rightarrow du = 2(x + 2) \, dx \Rightarrow \frac{1}{2} du = (x + 2) \, dx \).

Substituting and evaluating,
\[
\int_1^3 \frac{x + 2}{x^2 + 4x + 7} \, dx = \frac{1}{2} \int_{u(1)}^{u(3)} \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 + 4x + 7| \bigg|_1^3
\]
\[
= \frac{1}{2} \ln(28) - \frac{1}{2} \ln(12) = \frac{1}{2} [\ln(28) - \ln(12)] = \frac{1}{2} \ln \left( \frac{28}{12} \right) = \frac{1}{2} \ln \left( \frac{7}{3} \right)
\]
Notice that in evaluating the definite integral in Example 4, we completed the entire substitution process and the evaluation of the indefinite integral before returning the limits of integration, 1 and 3, to the problem. Here is an alternative to this technique:

**Example 5**: Determine the value of \( \int_{1}^{3} \frac{x + 2}{x^2 + 4x + 7} \, dx \).

Let \( u = x^2 + 4x + 7 \).

Express the limits of integration in terms of \( u \):
When \( x = 1, u = 12 \); when \( x = 3, u = 28 \).

Then \( \frac{du}{dx} = 2x + 4 \Rightarrow du = 2(x + 2) \, dx \Rightarrow \frac{1}{2} \, du = (x + 2) \, dx \).

Substituting and evaluating, with new limits of integration in terms of \( u \):
\[
\int_{1}^{3} \frac{x + 2}{x^2 + 4x + 7} \, dx = \frac{1}{2} \int_{12}^{28} \frac{du}{u} = \frac{1}{2} \ln u \bigg|_{12}^{28} = \frac{1}{2} \ln (28) - \frac{1}{2} \ln (12) = \frac{1}{2} [\ln (28) - \ln (12)] = \frac{1}{2} \ln \left( \frac{28}{12} \right) = \frac{1}{2} \ln \left( \frac{7}{3} \right)
\]

Here are examples using the last two integrals in the table developed previously. Example 6 is an indefinite integral and Example 7 is a definite integral.

**Example 6**: Evaluate \( \int x \ln \left( \frac{x^2 + 1}{x^2 + 1} \right) \, dx \).

Let \( u = \ln \left( \frac{x^2 + 1}{x^2 + 1} \right) \).

Then \( \frac{du}{dx} = \frac{1}{x^2 + 1} \cdot 2x \Rightarrow du = \frac{2x}{x^2 + 1} \, dx \) or \( \frac{1}{2} \, du = \frac{x}{x^2 + 1} \, dx \).

Substituting, \( \int \frac{x \ln (x^2 + 1)}{x^2 + 1} \, dx = \int u \cdot \frac{1}{2} \, du = \frac{1}{2} \int u \, du \),
and \( \frac{1}{2} \int u \, du = \frac{1}{2} \left( \frac{u^2}{2} + C \right) = \frac{1}{4} \left[ \ln (x^2 + 1) \right]^2 + C \).

Therefore, \( \int \frac{x \ln (x^2 + 1)}{x^2 + 1} \, dx = \frac{1}{4} \left[ \ln (x^2 + 1) \right]^2 + C \).
Example 7: Determine the value of \( \int_{0}^{2} \frac{5x^2}{e^{x^3}} \, dx \).

Let \( u = x^3 \).
Then \( \frac{du}{dx} = 3x^2 \Rightarrow du = 3x^2 \, dx \Rightarrow \frac{5}{3} \, du = 5x^2 \, dx \).
Substituting and evaluating,
\[
\int_{0}^{2} \frac{5x^2}{e^{x^3}} \, dx = \frac{5}{3} \int_{0}^{2} \frac{du}{e^{u}} = -\frac{5}{3} e^{-u} \bigg|_{0}^{2} + C = -\frac{5}{3} e^{-x^3} \bigg|_{0}^{2}
\]
\[
= -\frac{5}{3} e^{-2^3} - \left( -\frac{5}{3} e^{-0^3} \right) = -\frac{5}{3} (e^{-8} - e^0) = -\frac{5}{3} (e^{-8} - 1)
\]