

Calculus II
MAT 146
Integration Applications: Average Value of a Function

Here we apply integration to the task of determining the average value of a function for a specified interval of independent-variable values. These ideas extend what we typically think about *average value*, moving from discrete and finite cases to the continuous case for which an infinite number of function values exist within a finite interval of input values.

We begin by connecting with our current perspective on average value.

Question: What was the average temperature between midnight and noon yesterday?

Our response really depends on what we know about yesterday's temperatures.

If all we know is that it was 65° F. at midnight and that it was 81° F. at noon, the average temperature is:

$$\frac{65+81}{2} = \frac{146}{2} = 73^\circ F$$

If we also know that it was 76° F. at 6 am, we can re-compute an average temperature:

$$\frac{65+76+81}{3} = \frac{222}{3} = 74^\circ F$$

What if we know hourly readings?

12 midnight	65° F	4 am	73° F	9 am	78° F
1 am	67° F	5 am	74° F	10 am	79° F
2 am	69° F	6 am	76° F	11 am	80° F
3 am	71° F	7 am	77° F	12 noon	81° F
		8 am	77° F		

$$\begin{aligned} \text{Average Temp} &= \frac{65+67+69+\cdots+80+81}{13} \\ &= \frac{967}{13} = \frac{\sum_{i=1}^{13} T_i}{13} = \frac{1}{13} \sum_{i=1}^{13} T_i \\ &= 74.385^\circ F \end{aligned}$$

where T_i represents the i th hourly temperature that we know we know.

What if we know temperatures every minute? Every second?

$$\begin{aligned}
\text{Average Temp} &= \frac{\sum_{i=1}^n T_i}{n} = \frac{1}{n} \sum_{i=1}^n T_i \\
&= \left(\frac{b-a}{b-a} \right) \cdot \frac{1}{n} \cdot \sum_{i=1}^n T_i && \text{multiply by } 1 = \frac{b-a}{b-a} \\
&= \left(\frac{1}{b-a} \right) \cdot \sum_{i=1}^n T_i \cdot \left(\frac{b-a}{n} \right) && \text{rearrange factors} \\
&= \left(\frac{1}{b-a} \right) \cdot \sum_{i=1}^n T_i \cdot \Delta t && \left(\frac{b-a}{n} \right) = \Delta t : \text{subinterval size}
\end{aligned}$$

So we're gathering more and more temperatures, each time dividing the sum of those temperatures by the number of temperatures we know. The temperatures span the time interval from time a to time b .

What if we let the number of temperatures we know grow without bound?

Take the limit of $\left(\frac{1}{b-a} \right) \cdot \sum_{i=1}^n T_i \cdot \Delta t$ as $n \rightarrow \infty$.

But the T_i can just be seen as as a function describing the temperature from time a to time b . Therefore, as $n \rightarrow \infty$, we can use a **definite integral** to process this sum, if it exists.

$$\lim_{n \rightarrow \infty} \left[\left(\frac{1}{b-a} \right) \cdot \sum_{i=1}^n T_i \cdot \Delta t \right] = \left(\frac{1}{b-a} \right) \cdot \int_a^b T(t) dt$$

Generalizing, we now have a mechanism for determining the average value of a function f that spans some continuous range of input values.

The average value of a function f , on $a \leq x \leq b$, with f continuous on that interval, is:

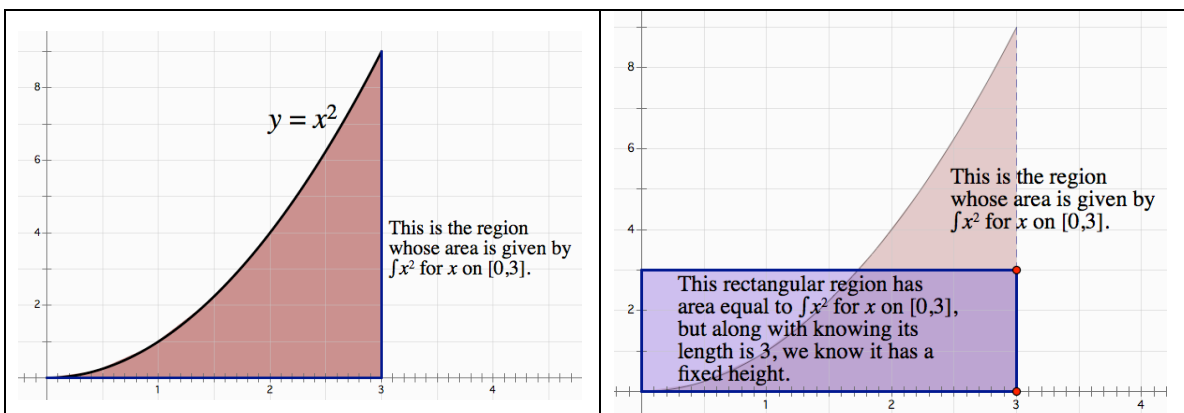
$$\frac{1}{b-a} \int_a^b f(x) dx$$

Example 1

Determine the average value of $y = x^2$ on $[0,3]$.

$$\bar{y} = \frac{1}{3-0} \int_0^3 x^2 dx = \frac{1}{3} \cdot \left. \frac{x^3}{3} \right|_0^3 = \frac{1}{3} \cdot 9 = 3$$

Let's look at this graphically as well.



The shaded region on the left has area $\int_0^3 x^2 dx$.

The rectangle on the right also has area $\int_0^3 x^2 dx$, and we know its length is 3.

So we have $\int_0^3 x^2 dx = 3 \cdot \text{height of rectangle}$.

We think of the rectangle's height as the *average value* of $y = x^2$ on $[0,3]$, because that height, or function value, is the same for all x on $[0,3]$. Then

$$\int_0^3 x^2 dx = 3 \cdot \text{height of rectangle} \Rightarrow \text{height of rectangle} = \frac{1}{3} \int_0^3 x^2 dx$$

So the average value of a function is just a rectangle's height, signifying the "spreading out" of the various function values over the interval so that at every location within the interval, the function takes on the same value.

If we represent that constant function value as $y = f(c)$, Then there has to exist some value c , $0 \leq c \leq 3$, that makes this average value work. The **Mean Value Theorem for Integrals** assures us that such a c exists, and we can, if needed, determine that value:

$$\int_0^3 x^2 dx = 3 \cdot \text{height of rectangle}$$

$$= 3 \cdot f(c) \Rightarrow f(c) = \frac{1}{3} \int_0^3 x^2 dx = 3. \quad \text{This is the average value of the function.}$$

$$\Rightarrow f(c) = 3 \Rightarrow c^2 = 3 \Rightarrow c = \sqrt{3}$$

This shows that when $x = \sqrt{3}$, we have a correct (average-value) rectangle height at $(\sqrt{3})^2 = 3$.

Example 2

Determine the average value of $y = 2x^4 - 3x$ for $-1 \leq x \leq 2$. Determine a value

$$c, -1 \leq c \leq 2, \text{ so that } f(c) = \frac{1}{(2 - (-1))} \int_{-1}^2 (2x^4 - 3x) dx = \frac{1}{3} \int_{-1}^2 (2x^4 - 3x) dx.$$

$$\begin{aligned}\bar{y} &= \frac{1}{2 - (-1)} \int_{-1}^2 (2x^4 - 3x) dx = \frac{1}{3} \int_{-1}^2 (2x^4 - 3x) dx \\ &= \frac{1}{3} \cdot \left(\frac{2}{5} x^5 - \frac{3}{2} x^2 \right) \Big|_{-1}^2 = \frac{1}{3} \cdot \left[\left(\frac{64}{5} - 6 \right) - \left(-\frac{2}{5} - \frac{3}{2} \right) \right] \\ &= \frac{1}{3} \left(\frac{68}{10} - \left(-\frac{19}{10} \right) \right) \\ &= \frac{29}{10}\end{aligned}$$

The average value of y on the stated interval is $29/10$. To determine c , $-1 \leq c \leq 2$, we now know $f(c) = 29/10$, so then

$$f(c) = \frac{29}{10} \Rightarrow 2c^4 - 3c = \frac{29}{10} \Rightarrow c \approx -0.753$$

Example 3

Return to the temperature example and suppose that $y(t) = 75 + 10 \sin\left(\frac{\pi}{12}t\right)$

represents the temperature at time t , where t is hours **after** 9 am. For (a) through (c), determine the average temperatures over each of the following intervals.

a) from 9 am to 9 pm	$\frac{1}{12} \int_0^{12} y(t) dt = 81.366^\circ F.$
b) from 10 am to 6 pm	$\frac{1}{8} \int_1^9 y(t) dt = 82.988^\circ F.$
c) from 8 am to 12 noon	$\frac{1}{4} \int_{-1}^3 y(t) dt = 77.472^\circ F.$
d) Over what 2-hour time period, measured on the hour, is the average temperature the warmest ?	Note that the graph of $y(t)$ reaches a maximum for $t = 6$ hrs (3 pm). Moving one hour in each direction from 3 pm, the warmest 2-hour period is from 2 pm to 4 pm.