The methods of substitution and integration by parts are widely used methods of integration. Each of these methods is associated with a derivative rule. Substitution relies on undoing the chain rule and integration by parts results from undoing the product rule. Additional methods of integration are associated with particular types of functions. Here, we explore how to integrate certain trigonometric functions.

**Trigonometric Integrals: Using Trig Identities**
Shown below are four examples to illustrate integration of certain families of trigonometric integrals.

**Example 1**: Evaluate \( \int \sin^3 x \, dx \).

\[
\int \sin^3 x \, dx \\
= \int (\sin x)(\sin^2 x) \, dx \\
= \int (\sin x)(1 - \cos^2 x) \, dx \\
= \int (\sin x - \sin x \cdot \cos^2 x) \, dx \\
= \int \sin x \, dx - \int (\sin x)(\cos^2 x) \, dx
\]

The first integral in the last line can be solved by inspection and the second by using the substitution \( u = \cos x \). With this in mind, the final result is

\[
\int \sin^3 x \, dx = -\cos x + \frac{1}{3} \cos^3 x + C
\]
Example 2: Evaluate $\int \cos^2 x \, dx$.

\[
\int \cos^2 x \, dx = \int \frac{1}{2}(1 + \cos(2x)) \, dx
\]
\[
= \frac{1}{2} \int (1 + \cos(2x)) \, dx
\]
\[
= \frac{1}{2} \int dx + \frac{1}{2} \int \cos(2x) \, dx
\]
\[
= \frac{1}{2} x + \frac{1}{2} \left( \frac{1}{2} \sin(2x) \right) + C
\]
\[
= \frac{1}{2} x + \frac{1}{4} \sin(2x) + C
\]

Example 3: Evaluate $\int \sin^5 x \, dx$.

\[
\int \sin^5 x \, dx = \int \sin x \left( \sin^2 x \right) \left( \sin^2 x \right) \, dx
\]
\[
= \int \sin x \left( 1 - \cos^2 x \right) \left( 1 - \cos^2 x \right) \, dx
\]
\[
= \int \sin x \left( 1 - 2 \cos^2 x + \cos^4 x \right) \, dx
\]
\[
= \int \left( \sin x - 2 \sin x \cos^2 x + \sin x \cos^4 x \right) \, dx
\]
\[
= \int \sin x \, dx - 2 \int \sin x \cos^2 x \, dx + \int \sin x \cos^4 x \, dx
\]

The left integral in the last line can be solved by inspection, while each of the other two require a $u$-substitution of $u = \cos x$. This gives us

\[
\int \sin^5 x \, dx = \int \sin x \, dx - 2 \int \sin x \cos^2 x \, dx + \int \sin x \cos^4 x \, dx
\]
\[
= - \cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C
\]
Example 4: Evaluate $\int \sin^4 x \, dx$.

\[
\int \sin^4 x \, dx = \int \sin^2 x \cdot \sin^2 x \, dx = \int \left[ \frac{1}{2}(1 - \cos(2x)) \right]^2 dx \\
= \frac{1}{4}(1 - 2\cos(2x) + \cos^2(2x)) \, dx = \frac{1}{4} \int dx - \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{4} \int \cos^2(2x) \, dx \\
= \frac{1}{4} \int dx - \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{4} \int \frac{1}{2} (1 + \cos(4x)) \, dx \\
= \frac{1}{4} \int dx - \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{8} \int (1 + \cos(4x)) \, dx \\
= \frac{1}{4} \int dx - \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{8} \int dx + \frac{1}{8} \int \cos(4x) \, dx
\]

Each of the four integrals in the last line can be solved by inspection or by a straightforward substitution. This gives us

\[
\int \sin^4 x \, dx = \frac{1}{4} \int dx - \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{8} \int dx + \frac{1}{8} \int \cos(4x) \, dx \\
= \frac{1}{4} x - \frac{1}{4} \sin(2x) + \frac{1}{8} x + \frac{1}{16} \sin(4x) + C \\
= \frac{3}{8} x - \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C
\]

As we look back on these four examples of integrals involving powers of trigonometric functions, we can make some useful observations about strategies for evaluating such integrals. With integrals of the form $\int \sin^n x \, dx$ or $\int \cos^n x \, dx$, where $n$ is a positive integer greater than 1, we follow one of two strategies depending on whether the exponent is odd or even.

Situation A: If $n = 2k + 1$ for $k$ some positive integer (i.e., $n$ is an odd exponent) [see Examples 1 and 3 above]:

1. Factor the integrand from $\int \sin^{2k+1} x \, dx$ to $\int \sin(x) \sin^{2k} x \, dx$.
2. Using the trig identity $\sin^2 x + \cos^2 x = 1$, rewrite $\int \sin(x) \sin^{2k} x \, dx$ as $\int \sin(x)(1 - \cos^2 x)^k \, dx$. 

3. Expand \((1 - \cos^2 x)^k\) to get
\[1 - k \cos^2 x + \cdots + (-1)^{k-1} k \cos^{2k-2} x + (-1)^k \cos^{2k} x.\]

4. Multiply each term in the above expansion by \(\sin x\) to get the integral
\[\int \sin x - k \sin x \cos^2 x + \cdots + (-1)^{k-1} k \sin x \cos^{2k-2} x + (-1)^k \sin x \cos^{2k} x \, dx.\]

5. Integrate this expression term by term:
\[\int \sin x \, dx - \int k \sin x \cos^2 x \, dx + \cdots + (-1)^{k-1} k \int \sin x \cos^{2k-2} x \, dx + (-1)^k \int \sin x \cos^{2k} x \, dx\]

6. Evaluate the first term by inspection and make a substitution in all other terms, using \(u = \cos x\).

7. Evaluate the remaining integral expressions.

Situation B: If \(n = 2k\) for \(k\) some positive integer (i.e., \(n\) is an even exponent) [see Examples 2 and 4 above]:

1. Using the trig identity \(\cos^2 x = \frac{1}{2}(1 + \cos(2x))\), rewrite \(\int \cos^{2k} x \, dx\) as
\[\int \left[\frac{1}{2}(1 + \cos(2x))\right]^k \, dx ,\]

2. Expand \(\left[\frac{1}{2}(1 + \cos(2x))\right]^k\) to get
\[1 + k \cos(2x) + \cdots + k \cos^{k-1}(2x) + \cos^k(2x),\]

3. Integrate the previous expression term by term to get
\[\left(\frac{1}{2}\right)^k \int dx + \left(\frac{1}{2}\right)^k \cdot k \int \cos(2x) \, dx + \cdots + \left(\frac{1}{2}\right)^k \cdot k \int \cos^{k-1}(2x) \, dx + \left(\frac{1}{2}\right)^k \int \cos^k(2x) \, dx\]

4. For each term in the previous sum, either (a) evaluate by inspection, (b) return to step (1) here and use the same trig identity (even exponents), or (c) return to Situation (A) previously described and follow those steps (odd exponents). Continue this process until each integral created can be solved by inspection.