Integral Tables

Our textbook contains more than 100 integral patterns in a table (see Reference Pages 4 through 8 at the back of the book), a small number compared to reference books devoted solely to integral tables.

Integral tables provide yet another way to help evaluate integrals. Using the tables, your goals should be to:

- **identify** patterns in the integral you are attempting to evaluate,
- **match** those patterns with one or more table entries,
- **create** any substitutions required to match the integral in question to one in the table, and
- **evaluate** the integral using the solutions provided within the chosen table entry and the substitutions you created.

Here are two examples.

**Example 1**: Evaluate \( \int \frac{1}{2 x^2 \sqrt{4 x^2 - 7}} dx \).

In this integral, a positive squared term appears inside and outside the radical and there is subtraction of a constant term inside the radical. If we multiply the numerator and denominator each by 4, giving us

\[
\int \frac{4}{2 \cdot 4 x^2 \sqrt{4 x^2 - 7}} dx = 4 \int \frac{dx}{2 \cdot 4 x^2 \sqrt{4 x^2 - 7}},
\]

this integral matches precisely the pattern shown in table entry #45 on Reference Page 5:

\[
\int \frac{du}{u^2 \sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C.
\]

To match, we need to make two substitutions:

- The constant 7 matches the position of \( a^2 \), so \( a^2 = 7 \). Note that the solution to the integral only requires \( a^2 \), so there is no need to determine that \( a = \sqrt{7} \), although in some cases that may be required.
Within the radical expression, \( u^2 = 4x^2 \), leading to \( u = 2x \) and \( du = 2dx \), or \( \frac{1}{2}du = dx \).

We now rebuild the integral in terms of \( u \) by making the substitutions just created, including restatement of the upper and lower limits of integration:

\[
4 \int_2^3 \frac{dx}{2x^2 \sqrt{4x^2 - 7}} = 4 \int_4^6 \frac{\frac{1}{2}du}{u^2 \sqrt{u^2 - a^2}} = 2 \int_4^6 \frac{du}{4u^2 \sqrt{u^2 - a^2}}.
\]

We now evaluate this integral using the pattern in the table entry:

\[
2 \int_4^6 \frac{du}{u^2 \sqrt{u^2 - a^2}} = 2 \left( \frac{u^2 - a^2}{a^2 u} \right) \bigg|_4^6 = 2 \left( \frac{\sqrt{29}}{42} - \frac{3}{28} \right) = \frac{2\sqrt{29} - 9}{42}.
\]

Example 2: Evaluate \( \int (1 + \ln x) \sqrt{1 + (x \ln x)^2} \, dx \).

If we let \( u = x \ln x \), then \( du = (\ln x + 1) \, dx \), and the integral can be rewritten as

\[
\int (1 + \ln x) \sqrt{1 + (x \ln x)^2} \, dx = \int \sqrt{1 + u^2} \, du
\]

This matches precisely the pattern in entry #21 on Reference Page 4, where \( a = 1 \) and \( u^2 = (x \ln x)^2 \):

\[
\int \sqrt{a^2 + u^2} \, du = \frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln \left( u + \sqrt{a^2 + u^2} \right) + C.
\]

All that remains is to substitute back into the right-side expression to get a solution:

\[
\frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln \left( u + \sqrt{a^2 + u^2} \right) + C = \frac{x \ln x}{2} \sqrt{1 + (x \ln x)^2} + \frac{1}{2} \ln \left( x \ln x + \sqrt{1 + (x \ln x)^2} \right) + C.
\]

Using CAS to Evaluate Integrals

Your TI-89 or TI-92 calculator (or an equivalent calculator or computer program) has within it a CAS, or computer algebra system. This means that you can perform symbolic operations with the calculator and in many cases get exact solutions in symbolic form. Such calculators will solve equations, factor expressions, make trigonometric substitutions, and a host of other algebraic and arithmetic manipulations. Your CAS can be used to evaluate definite and indefinite integrals as well. Return to Examples 1 and 2 above and attempt to solve these using your CAS.
You will find that Example 1 produces the same numerical solution as we did. Example 2, however, is problematic for the TI-89. When we enter the original integral shown in Example 2,
\[ \int (1 + \ln x) \sqrt{1 + (x \ln x)^2} \, dx, \]
the TI-89 returns the following:
\[ \int \left( \ln x \sqrt{x^2 (\ln x)^2 + 1} \right) dx + \int \left( \sqrt{x^2 (\ln x)^2 + 1} \right) dx. \]
The only action taken by the TI-89 is to expand the integral into the sum of two integrals and to expand a squared term within the radical. It has not carried out any evaluation of the integral.\(^1\)

The moral of that story is that a CAS can be very helpful, but it may not always produce a solution for you or it may not produce a solution as efficiently or compactly as you may do with pencil and paper and the use of tables or other integration techniques.

\(^1\) If you enter \( \int \sqrt{a^2 + u^2} \, du \) into the TI-89, it will generate the same evaluation as shown in table entry #21 (with the constant of integration missing), so the TI-89 does know table patterns!