Again we use a definite integral to sum an infinite number of measures, each infinitesimally small. We seek to determine the length of a curve that represents the graph of some real-valued function \( f \), measuring from the point \((a, f(a))\) on the curve to the point \((b, f(b))\) on the curve.

The graph of \( y = f \) is shown. We seek to determine the length of the curve, known as arc length, from the point \((a, f(a))\) on the curve to the point \((b, f(b))\).

Together with endpoints \((a, f(a))\) and \((b, f(b))\), we select additional points \((x_i, f(x_i))\) on the curve. Here, we’ve created \( n \) line segments. The length of the \( i \)th line segment is denoted by \( L_i \), with \( 1 \leq i \leq n \).

Concentrate on the segment whose length is \( L_i \). Determine the slope of that segment:
slope of segment whose length is $L_i = m_i$

\[
L_i = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}
\]

We know by the Mean Value Theorem that there exists an $x_{i*}$, with $x_{i-1} \leq x_{i*} \leq x_{i-1}$, such that $m_i = f'(x_{i*})$. We use this fact, and the distance formula, to calculate the typical segment length $L_i$:

\[
L_i = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}
= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}
= \sqrt{(\Delta x_i)^2 + (m_i \Delta x_i)^2}
= \sqrt{(\Delta x_i)^2 + (f'(x_{i*}) \Delta x_i)^2}
= \sqrt{(\Delta x_i)^2 \left[1 + \left(f'(x_{i*})\right)^2\right]}
= \Delta x_i \sqrt{1 + \left(f'(x_{i*})\right)^2}
\]

We now let the number of segments between $(a, f(a))$ and $(b, f(b))$ grow without bound. That is, we let $n \to \infty$. We can now write a definite integral to add all the lengths $L_i$:

\[
total \ length \ of \ the \ curve \ from \ x = a \ to \ x = b : \int_a^b \sqrt{1 + \left[f'(x)\right]^2} \ dx
\]

**Example #1**: Determine the length of the curve $y = x^2$ from $x = 0$ to $x = 3$.

We use the arc length formula $\int_a^b \sqrt{1 + \left[f'(x)\right]^2} \ dx$. Here, $a = 0$ and $b = 3$. The function $y = f(x) = x^2$ has derivative $f'(x) = 2x$, so
\[
arc length = \int_a^b \sqrt{1 + \left[f'(x)\right]^2} \, dx
\]
\[
= \int_0^3 \sqrt{1 + [2x]^2} \, dx
\]
\[
= \int_0^3 \sqrt{1 + 4x^2} \, dx
\]
\[
= \frac{1}{4} \ln(\sqrt{37} + 6) + \frac{3}{2} \sqrt{37}
\]
\[
\approx 9.747089
\]

**Example #2:** What is the arc length for 
\( y = \frac{x^3}{6} + \frac{1}{2x}, \frac{1}{2} \leq x \leq 1 \)?

Again, use the arc length formula 
\[
\int_a^b \sqrt{1 + \left[f'(x)\right]^2} \, dx
\]. Here, \( a = \frac{1}{2} \) and \( b = 1 \), with

\[
\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}.
\]

\[
arc length = \int_{\frac{1}{2}}^1 \sqrt{1 + \left[\frac{x^2}{2} - \frac{1}{2x^2}\right]^2} \, dx
\]
\[
= \frac{31}{48}
\]

**Example #3:** For \( x = \ln(1 - y^2) \), determine the length of arc \( x(y) \) for \( 0 \leq y \leq \frac{1}{2} \).

Here, we use the arc length formula with \( y \) as the independent variable: 
\[
\int_c^d \sqrt{1 + [h'(y)]^2} \, dy.
\]

Here, we have \( c = 0 \) and \( d = \frac{1}{2} \), with \( \frac{dx}{dy} = \left(\frac{1}{1 - y^2}\right)(-2y) = \frac{-2y}{1 - y^2} \).

\[
arc length = \int_{0}^{\frac{1}{2}} \sqrt{1 + \left[-\frac{2y}{1 - y^2}\right]^2} \, dy
\]
\[
= 0.5986123
\]