What is a Differential Equation? Also known as “DIF-E-Q” or “D.E.”

- An equation in which a rate of change is expressed in terms of an independent and dependent variable.

**Examples of Differential Equations**

<table>
<thead>
<tr>
<th>$\frac{dy}{dx}$</th>
<th>$\frac{dP}{dt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x + 5y$</td>
<td>$K \cdot P \left(1 - \frac{P}{100}\right)$</td>
</tr>
<tr>
<td>$C \cdot y$</td>
<td>$y' = 2x^2 - 8y + y^2$</td>
</tr>
<tr>
<td>$y'' + y' = 2y + 5x^2$</td>
<td></td>
</tr>
</tbody>
</table>

You already know how to solve a huge number of differential equations.

i) If $\frac{dy}{dx} = 3x^2 - 8x + 1...$

   Then $y = x^3 - 4x^2 + x + C$

   This is a **solution** to that differential equation, found by determining the anti-derivative of $\frac{dy}{dx}$.

   This is a **differential equation**, because it expresses a rate of change, $\frac{dy}{dx}$, in terms of the independent variable $x$.

ii) $y' = \frac{1}{x}$  $\Rightarrow$  $y = \ln|x| + C$

iii) $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ $\Rightarrow$  $y = \sin^{-1}(x) + C$

Here are some everyday situations where rate of change is imbedded within a relationship.

**Compound interest: “interest on interest”**

$$A(t) = P \left(1 + \frac{r}{n}\right)^{t\cdot n}$$

where $A(t)$ is the account value at time $t$ in years, $P$ is the principle, $r$ is the annual interest rate, and $n$ is the number of compounding periods per year.

**Population Growth: future population depends on the current population**

$$P(t) = P_0 (1 + r)^t$$

where $P(t)$ is the population at time $t$, in some units of time $t$, $P_0$ is the initial population, and $r$ is the growth rate per unit of time.
What is a solution to a differential equation?

- A **general solution** is a family of functions that satisfies a given differential equation.
- A **particular solution** (also called the solution to an initial-value problem) is a particular function that satisfies both a given differential equation and some specified ordered pair for the function.

Example 1

Show that $y = x - \frac{1}{x}$ is a solution to the differential equation $xy' + y = 2x$.

Step 1: If $y = x - \frac{1}{x}$ then $\frac{dy}{dx} = y' = 1 + \frac{1}{x^2}$.

Step 2: Substitute this known representation for $y'$ into the original differential equation.

$$xy' + y = 2x$$

$$x\left(1 + \frac{1}{x^2}\right) + \left(x - \frac{1}{x}\right) = 2x$$

Step 3: Show that the left-side expression (LS) is equivalent to the right-side expression (RS).

$$LS = x\left(1 + \frac{1}{x^2}\right) + \left(x - \frac{1}{x}\right)$$

$$= \left(x + \frac{1}{x}\right) + \left(x - \frac{1}{x}\right)$$

$$= 2x$$

$$= RS$$

We’ve shown that the LS expression is equivalent to the RS expression, so we know $y = x - \frac{1}{x}$ is a solution to the differential equation $xy' + y = 2x$.

Example 2

Show that $y = (\sin x)(\cos x) - \cos x$ is a solution to the initial-value problem (IVP)

$$y' + (\tan x)y = \cos^2 x$$ with $y(0) = -1$ on $-\frac{\pi}{2} < x < \frac{\pi}{2}$
Step 1: If \( y = (\sin x)(\cos x) - \cos x \), we know that \( y' = \cos^2 x - \sin x(\sin x - 1) \).

Step 2: Into the original DE, sub in the known expression for \( y \) as well as the known expression for its derivative, \( y' \), and then show that the DE is true.

For
\[
y' + (\tan x)y = \cos^2 x \tag{1},
\]
with
\[
y = (\sin x)(\cos x) - \cos x \quad \text{and} \quad y' = \cos^2 x - \sin x(\sin x - 1),
\]
can we show that
\[
LS: \quad y' + (\tan x)y
\]
is equivalent to
\[
RS: \quad \cos^2 x ?
\]
\[
LS = y' + (\tan x)y
= \left[\cos^2 x - \sin x(\sin x - 1)\right] + (\tan x)\left[(\sin x)(\cos x) - \cos x\right]
= \cos^2 x - \sin^2 x + \sin x + (\tan x)(\sin x)(\cos x) - (\tan x)(\cos x)
= \cos^2 x - \sin^2 x + \sin x + \left(\frac{\sin x}{\cos x}\right)(\sin x)(\cos x) - \left(\frac{\sin x}{\cos x}\right)(\cos x)
= \cos^2 x - \sin^2 x + \sin x + \sin^2 x - \sin x
= \cos^2 x
= RS
\]
This shows that the LS expression of (1) above is equivalent to the RS expression. This shows that \( y = (\sin x)(\cos x) - \cos x \) is a solution to \( y' + (\tan x)y = \cos^2 x \).

We also must show that \( y(0) = -1 \):
\[
y = (\sin x)(\cos x) - \cos x
y(0) = (\sin 0)(\cos 0) - \cos 0
= (0)(1) - 1
= -1
\]