MAT 146
Direction Fields: Graphical Representations for Differential Equations

9.2

If we cannot analytically determine a solution to a differential equation, what options do we have for generating a solution? Similar to our efforts in evaluating integrals—we found many functions whose exact antiderivative could not be determined—we have developed techniques that generate close approximations to solutions to differential equations.

Here, we describe and illustrate an approximation technique that generates a graphical representation called a **slope field** or a **direction field**. In a related discussion, we describe a numerical approximation technique call **Euler's Method**. Leonard Euler was a prolific Swiss mathematician. His last name is pronounced “Oil-er.”

**Example 1**

Solve the differential equation \( xy' + y = 2x \).

**Step 1**

Re-express the differential equation by solving for \( y' \) if possible.

\[
x y' + y = 2x
\]

\[
y' = \frac{2x - y}{x}
\]

\[
y = 2 - \frac{y}{x}
\]

**Step 2**

Calculate slopes—particular values of \( y' \)—for specified ordered pairs \((x,y)\).

<table>
<thead>
<tr>
<th>Selected ordered pairs ((x,y))</th>
<th>Corresponding slopes (y' = 2 - \frac{y}{x})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0) (y \in \mathbb{R})</td>
<td>(y' = 2 - \frac{y}{0} \Rightarrow) No Slope!</td>
</tr>
<tr>
<td>(1) (0)</td>
<td>(y' = 2 - \frac{0}{1} = 2)</td>
</tr>
<tr>
<td>(2) (0)</td>
<td>(y' = 2 - \frac{0}{2} = 2)</td>
</tr>
<tr>
<td>(x \neq 0) (0)</td>
<td>(y' = 2 - \frac{0}{x} = 2)</td>
</tr>
<tr>
<td>(1) (1)</td>
<td>(y' = 2 - \frac{1}{1} = 1)</td>
</tr>
<tr>
<td>(1) (2)</td>
<td>(y' = 2 - \frac{2}{1} = 0)</td>
</tr>
<tr>
<td>(1) (-1)</td>
<td>(y' = 2 - \frac{-1}{1} = 3)</td>
</tr>
<tr>
<td>(1) (-2)</td>
<td>(y' = 2 - \frac{-2}{1} = 4)</td>
</tr>
</tbody>
</table>
Step 3
On an xy-coordinate system, locate each ordered pair from your table and graph the associated slope using a hash mark.

This is called a **slope field** or a **direction field**. We can generate direction fields for many differential equations using the TI-89 and other graphing devices.

Here are two more complete version of the slope field for

\[ y' = 2 - \frac{y}{x} \]

The first uses hash marks at the same level as the one above, potted at integer-valued ordered pairs. The second uses the same window but has a much greater concentration of hash marks within that same window.

To graph a solution to the original differential equation, choose one ordered pair as an **initial value**. Plot the slope at that point, if not already shown with a hash mark, and then trace a path for the solution curve using the existing hash marks in the slope field as a guide.

The curve superimposed on the slope field here is a graphical approximation to the solution to the differential equation

\[ xy' + y = 2x \]

with initial value (1,3) satisfying \( y \).
Example 2

For the differential equation $2y' + 6y = x^2$:

a) Create a slope field on the window $-2 \leq x \leq 6, -2 \leq y \leq 6$.

b) Sketch graphs of at least four different members of the general family of solutions to this differential equation.

c) Identify the graph of the particular solution for the initial condition that $y = 1$ when $x = 0$.

Solutions

a) Look at the first plot shown here. Remember to first solve the differential equation for $y'$.

b) The second plot here shows solution graphs for various initial conditions, including $(1,3), (0,1), (0,-1), (-2,-2)$, and $(-2,2)$.

c) Look for the initial condition, $(0,1)$, labeled on the second plot shown here.