More Trig Integrals!

#1) $\int 3\cos^3(x)\sin^2(x)\,dx$	#2) $\int \cos^3(2x)\sin^3(2x)\,dx$
#3) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 3\sin^2(\theta)\,d\theta$	#4) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 2\sec^2(u)\,du$
#5) $\int 2\tan^3(u)\cos^2(u)\,du$	#6) $\int \tan^3(x)\sec(x)\,dx$
#7) $\int \sec(x)\,dx$	

MORE TRIG INTEGRALS (7.2)

#1) $\int 3\cos^3 x \sin^2 x \, dx = 3\int (1-\sin^2 x)\sin^2 x \cos x \, dx$

Let $u = \sin x$ so $du = \cos x \, dx$ $\Rightarrow$

$$3\int (1-\sin^2 x)\sin^2 x \cos x \, dx = 3\int (1-u^2)u^2 \, du$$

$$= 3\int u^2 - u^4 \, du = 3\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C$$

$$= u^3 - \frac{3}{5}u^5 + C = \sin^3 x - \frac{3}{5}\sin^5 x + C$$

#2) $\int \cos^3(2x)\sin^3(2x) \, dx = \int \cos^2(2x)\sin^3(2x)\cdot\cos(2x) \, dx$

$= \int (1-\sin^2(2x))\sin^3(2x)\cos(2x) \, dx$; let $u = \sin(2x)$

$du = 2\cos(2x)dx$

$\Leftrightarrow \frac{1}{2}du = \cos(2x)dx$

$$= \frac{1}{2}\int (1-u^2)u^3 \, du$$

$$= \frac{1}{2}\int u^3 - u^5 \, du = \frac{1}{2}\left(\frac{1}{4}u^4 - \frac{1}{6}u^6\right) + C$$

$$= \frac{1}{8}\sin^4(2x) - \frac{1}{12}\sin^6(2x) + C$$

#3) $\displaystyle\int_{\pi/4}^{\pi/3} 3\sin^2\theta\, d\theta = 3\int_{\pi/4}^{\pi/2} \frac{1-\cos(2\theta)}{2}\, d\theta$

$$= \frac{3}{2}\int_{\pi/4}^{\pi/3} 1-\cos(2\theta)\, d\theta = \frac{3}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_{\pi/4}^{\pi/3}$$

$$= \frac{3}{2}\left(\pi/3 - \frac{1}{2}\sin\left[2\cdot\frac{\pi}{3}\right]\right) - \frac{3}{2}\left(\pi/4 - \frac{1}{2}\sin(2\cdot\pi/4)\right)$$

$$= \frac{\pi}{2} - \frac{3}{4}\sin\left(\frac{2\pi}{3}\right) - \frac{3\pi}{8} + \frac{3}{4}\sin(\pi/2)$$

$$= \frac{\pi}{8} - \frac{3}{4}\left(\frac{\sqrt{3}}{2}\right) + \frac{3}{4}(1)$$

$$= \frac{\pi}{8} - \frac{3}{8}\sqrt{3} + \frac{3}{4} = \frac{1}{8}\left(6 + \pi - 3\sqrt{3}\right)$$

#4) $\displaystyle\int_{\pi/6}^{\pi/4} 2\sec^2 u\,du = 2\tan u\Big]_{\pi/6}^{\pi/4}$

$= 2\tan \pi/4 - 2\tan \pi/6 = 2(1) - 2\left(\frac{\sqrt{3}}{3}\right) = 2 - \frac{2}{3}\sqrt{3}$

#5) $\displaystyle\int 2\tan^3 u\cos^2 u\,du = \int 2\tan^2 u\tan u\cos^2 u\,du$

$= 2\displaystyle\int (\sec^2 u - 1)\cdot\frac{\sin u}{\cos u}\cdot\cos^2 u\,du$

$= 2\displaystyle\int \left(\frac{1}{\cos^2 u} - 1\right)\sin u\cos u\,du$

$= 2\displaystyle\int \frac{\sin u\cos u}{\cos^2 u} - \sin u\cos u\,du$

$= 2\displaystyle\int \frac{\sin u}{\cos u} - \sin u\cos u\,du$

$= 2\displaystyle\int \left(\frac{1}{\cos u} - \cos u\right)\sin u\,du$; let $v = \cos u$, $dv = -\sin u\,du$

$= -2\displaystyle\int \left(\frac{1}{v} - v\right)dv = -2\left(\ln|v| - \frac{1}{2}v^2\right) + C$

$= v^2 - 2\ln|v| + C = \cos^2 u - 2\ln|\cos u| + C$

#6) $$\int \tan^3 x \sec x \, dx = \int \frac{\sin^3 x}{\cos^3 x} \cdot \frac{1}{\cos x} \, dx$$

$$= \int \frac{\sin^3 x \, dx}{\cos^4 x} = \int \frac{(1-\cos^2 x)\sin x \, dx}{\cos^4 x}$$

let $u = \cos x$

$du = -\sin x \, dx$

$$\Rightarrow \int \frac{(1-\cos^2 x)\sin x \, dx}{\cos^4 x} = -\int \frac{(1-u^2)}{u^4} \, du$$

$$= -\int \frac{1}{u^4} - \frac{1}{u^2} \, du = \int u^{-2} - u^{-4} \, du$$

$$= \frac{u^{-1}}{-1} - \frac{u^{-3}}{-3} + C = -\frac{1}{u} + \frac{1}{3u^3} + C$$

$$= -\frac{1}{\cos x} + \frac{1}{3\cos^3 x} + C = \frac{1}{3}\sec^3 x - \sec x + C$$

#7) $\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

let $u = \sec x + \tan x$

so $du = \sec x \tan x + \sec^2 x \, dx$

$\Rightarrow \int \frac{\sec^2 x + \sec x \tan x \, dx}{\sec x + \tan x} = \int \frac{du}{u}$

$= \ln|u| + C = \ln|\sec x + \tan x| + C$